Myopic Management and Economic Instability*

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Abstract

This paper explores the relationship between economic instability and myopic management. A manager is said to be forward looking if they maximize the present value of their firm's future profit over an infinite horizon. Conversely, a manager is said to be myopic if they maximize the present value of their firm's future profit over a finite horizon. If managers are forward looking then output and employment are shown to converge on the steady state as quickly as possible following an unanticipated shock. In contrast, myopic management is shown to amplify unanticipated shocks and produce endogenous deviations from the steady state. Sufficiently active monetary policy is shown to stabilize output and employment on the steady state by incentivizing myopic mangers to adopt forward looking strategies.

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1 Introduction

This paper explores the relationship between economic instability and myopic management in markets where firms employ labor and capital to produce homogeneous output and interest rates depend on monetary policy. Each firm has a manager who directs its activity. Managers are said to be forward looking if they aim to maximize the present value of their firm's future profit over an infinite horizon. Conversely, managers are said to be myopic if they aim to maximize the present value of their firm's future profit over a finite horizon. Firms can freely adjust their labor employment, but there is an upper bound on the rate at which firms can adjust capital employment.

If managers are perfectly forward looking then output and employment are shown to converge on the steady state as quickly as possible following an unanticipated shock. In contrast, myopic managers are shown to amplify unanticipated shocks by overadjusting output and employment beyond their steady state values. Firms own the physical capital they employ, so the user cost of capital depends on both the opportunity cost of holding capital and the rate of change in the price of capital. Changes in the price of capital assets can distort the user cost of capital, leading to booms and busts that overshoot the steady state. Decreasing capital prices temporarily raise the user cost of capital, incentivizing myopic managers to underemploy capital. Conversely, increasing capital prices temporarily lower the user cost of capital, incentivizing myopic managers to overemploy capital.

Sufficiently active monetary policy is shown to stabilize output and employment on the steady state regardless of whether managers are myopic or forward looking. Consistent with conventional wisdom, this policy lowers interest rates when output is below the steady state and raises interest rates when output is above the steady state. It incentivizes myopic mangers to adopt forward looking strategies by counteracting distortions to the user cost of capital. When output is below the steady state, this policy lowers interest rates enough to keep the user cost of capital below the return on capital. When output is above the steady state, it raises interest rates enough to keep the user cost of capital above the return on capital.

This paper contributes to the literature investigating the mechanisms that drive economic instability (Beaudry, Galizia, and Portier, 2020; Sufi and Taylor, 2021). The results of the present paper contrast with those of Gali (2014) who finds that higher interest rates produce larger bubbles in overlapping generations models with nominal rigidities. In contrast, the present paper finds that higher interest rates can prevent output, employment, and capital prices from the exceeding their steady state values. Firms are assumed to own the capital they employ as in the model presented by Carceles-Poveda and Coen-Pirani (2010). Cooper and Haltiwanger (2006) and Wang and Wen (2012) consider capital adjustment costs, while the present paper considers bounded capital adjustment rates.

The remainder of the paper is organized as follows. Section 2 describes the model, section 3 presents the results, and section 4 concludes. Proofs are provided in the appendix.

2 Model

We consider a market where a continuum of firms $i \in [0, 1]$ employ labor and capital to produce homogeneous output. Time is indexed by $t \in \mathbb{R}_+$. Let $\ell_i(t)$ denote the quantity of labor employed by firm i at time t. Let $k_i(t)$ denote the quantity of capital employed by firm i at time t. The production technology exhibits constant returns to scale and constant elasticity in each factor of production. Let $y_i(t)$ denote firm i's output at time t.

$$y_i(t) = A\ell_i(t)^a k_i(t)^{1-a} \tag{1}$$

Here A denotes total factor productivity and $a \in (0, 1)$ denotes the output elasticity of labor. Let $\dot{k}_i(t)$ denote the right-derivative of $k_i(t)$. The rate at which firms can adjust their capital employment over time is bounded by $\chi > 0$. This constraint formalizes the idea that capital is fixed in the short run but variable in the long run. Let $x_i(t) \in [-\chi, \chi]$ denote the rate at which firm i adjusts its capital employment at time t.

$$\dot{k}_i(t) = x_i(t) \tag{2}$$

Firm i's capital employment $k_i(t)$ is assumed to be continuous and right-differentiable in t. The capital adjustment rate $x_i(t)$ is assumed to be right-continuous in t. Let $X(t)$ denote the rate of change in total capital employment at time t.

$$\dot{K}(t) = X(t) = \int_0^1 x_i(t)\,di \tag{3}$$

Let $r(t)$ denote the real interest rate at time t. Let $R(t,s)$ denote the growth of an interest-bearing deposit over the closed interval $[t,s]$.

$$R(t,s) = \exp\left(\int_t^s r(\tau)\,d\tau\right) \tag{4}$$

Let $Y(t)$ denote total output at time t.

$$Y(t) = \int_0^1 y_i(t)\,di \tag{5}$$

Let $L(t)$ denote total labor employment at time t.

$$L(t) = \int_0^1 \ell_i(t)\,di \tag{6}$$

The real wage $w(t)$ is assumed to be an increasing convex isoelastic function of labor employment.

$$w(t) = BL(t)^b \tag{7}$$

Let $K(t)$ denote total capital employment at time t.

$$K(t) = \int_0^1 k_i(t)\,di \tag{8}$$

The real price of capital $p(t)$ is assumed to be an increasing convex isoelastic function of capital employment.

$$p(t) = QK(t)^{\kappa} \tag{9}$$

2.1 Profit

Firms own the capital they employ, so the user cost of capital $c(t)$ is equal to the opportunity cost of capital less the rate of change in the price of capital.

$$c(t) = r(t)\,p(t) - \dot{p}(t) \tag{10}$$

The opportunity cost of capital $r(t)\,p(t)$ is the return that could be earned from an interest bearing deposit of $p(t)$ at time t. The right-derivative $\dot{p}(t)$ is the rate of change in the real price of physical capital at time t. Firms are assumed to own the physical capital they employ, so the user cost of capital is lower when rate of change in the price of capital is positive and the user cost of capital is higher when the rate of change in the price of capital is negative. The price of output is normalized to one. Firm i's profit $\pi_i(t)$ is equal to its sales revenue less its labor cost and its capital cost.

$$\pi_i(t) = y_i(t) - w(t)\,\ell_i(t) - c(t)\,k_i(t) \tag{11}$$

Firms can freely adjust their labor employment and aim to maximize profit, so the first order condition on firm i's labor employment says that the wage is equal to the marginal product of labor.

$$w(t) = aA\ell_i(t)^{a-1}\,k_i(t)^{1-a} \tag{12}$$

Proposition 1 says that total output and total labor employment are increasing concave isoelastic functions of total capital employment. Higher capital employment coincides with higher labor employment since the marginal product of labor is increasing in capital employment.

Proposition 1. *There exists $G_L, G_Y > 0$ such that for all $h \in H$*

$$L(t) = G_L K(t)^{\gamma_L} \qquad Y(t) = G_Y K(t)^{\gamma_Y} \tag{13}$$

$$\gamma_L = \frac{1-a}{1-a+b} \qquad \gamma_Y = 1 - a + \gamma_L a \tag{14}$$

Proposition 1 says that firm i's profit at time t is linear in its capital employment at time t. The excess return on capital $\alpha(t)$ is equal to the return on capital $GK(t)^{-\gamma}$ less the user cost of capital $c(t)$. If the excess return on capital is positive at time t then firm i's profit at time t is increasing in its capital employment at time t. If the excess return on capital at time t is negative then firm i's profit at time t is decreasing in its capital employment at time t.

Proposition 2. *There exists $G > 0$ such that*

$$\pi_i(t) = \alpha(t) k_i(t) \tag{15}$$

$$\alpha(t) = GK(t)^{-\gamma} - c(t) \tag{16}$$

$$\gamma = \frac{ab}{1-a+b} \tag{17}$$

2.2 Monetary Policy

Let H denote the set of all possible paths $h : \mathbb{R}_+ \to \mathbb{R}^5_+$ such that

$$h(t) = (Y(t), L(t), K(t), w(t), p(t)) \tag{18}$$

Let h^t denote the history of the path $h \in H$ up to time $t \in \mathbb{R}_+$. More formally $h^t : [0,t] \to \mathbb{R}^5_+$ denotes the restriction of the path $h \in H$ to the interval $[0,t]$ such that $h^t(s) = h(s)$ for all $s \in [0,t]$. Let $\mathcal{H}$ denote the set of all possible histories.

$$\mathcal{H} = \left\{h^t : h \in H, t \in \mathbb{R}_+\right\} \tag{19}$$

The real interest rate $r(t)$ is determined by the monetary policy $\varphi : \mathcal{H} \to \mathbb{R}$ as a function of the history h^t such that $r(t) = \varphi(h^t)$. Let $\overline{\varphi} : \mathcal{H} \to \mathbb{R}$ denote the monetary policy that maintains a constant real interest rate $\overline{r} \in \mathbb{R}$ such that $\overline{\varphi}(h^t) = \bar{r}$ for all $h^t \in \mathcal{H}$.

2.3 Equilibrium

Each firm has a manager who directs its activity. Let $\sigma_i^t : \mathcal{H} \to [-\chi, \chi]$ denote manager i's strategy at time t such that $x_i(t) = \sigma_i^t(h^t)$. Let Σ_i denote the set of all possible management strategies. Let $\Sigma = \prod_{i \in [0,1]} \Sigma_i$ denote the set of all possible strategy profiles. Let $\beta \in \mathbb{R}_{++}$ denote the management horizon. In equilibrium, manager i selects $\sigma_i^t \in \Sigma_i$ to maximize firm i's total discounted profit from time t time $t + \beta$ given firm i's current capital employment $k_i(t)$, the history h^t, and $x_j(s) = \sigma_j^t(h^s)$ for $j \neq i$ and $s \in [t, t + \beta]$.

$$\sigma_i^t \in \underset{\sigma_i^t \in \Sigma_i}{\arg\max} \int_t^{t+\beta} R(s,t)\, \pi_i(s) \tag{20}$$

Let $\Sigma^*(h^t)$ denote the set of all strategy profiles that satisfy the equilibrium condition (20) under the history $h^t \in \mathcal{H}$. Let H^* denote the set of all equilibrium paths $h \in H$ such that for all $t \in \mathbb{R}_+$ there exists $\sigma^t \in \Sigma^*(h^t)$ such that

$$\dot{K}(t) = \int_0^t \dot{k}_i(t)\, dt = \int_0^t x_i(t)\, dt = \int_0^t \sigma_i^t\left(h^t\right) dt \tag{21}$$

A state vector $\bar{h} \in \mathbb{R}_+^5$ is said to be a steady state if there exists an equilibrium path $h \in H^*$ and a time $t \in \mathbb{R}_+$ such that $h(s) = \bar{h}$ for $s \geq t$. Proposition 3 characterizes the steady state under the monetary policy $\overline{\varphi}$.

Proposition 3. *If $\varphi = \bar{\varphi}$ and $\bar{h} = \left(\bar{Y}, \bar{L}, \bar{K}, \bar{w}, \bar{p}\right) \in \mathbb{R}_+^5$ is a steady state then*

$$\bar{L} = G_L \bar{K}^{\gamma_L} \qquad \bar{K} = \left(\frac{G}{Q\bar{r}}\right)^{\frac{1}{\kappa+\gamma}} \qquad \bar{Y} = A\bar{L}^a \bar{K}^{1-a} \tag{22}$$

3 Results

Theorem 1 characterizes the equilibrium path under forward looking management and constant real interest rates. Equation (23) says that the equilibrium path converges on the steady state following an unanticipated shock at time $t = 0$. Equation (24) says that the distance to the steady state always decreases over time, so the path never moves away from the steady state. Equation (24) says that the path moves toward the steady state as quickly as possible since the rate at which firms can adjust their capital employment over time is bounded by χ.

Theorem 1. *If $\beta = \infty$ and $\varphi = \overline{\varphi}$ and $h \in H^*$ then for all $t \in \mathbb{R}$*

$$\lim_{t \to \infty} h(t) = \bar{h} \tag{23}$$

$$\frac{d}{dt} \left| h(t) - \bar{h} \right| < 0 \tag{24}$$

$$\dot{K}(t) = \chi \left(\bar{K} - K(t) \right) \tag{25}$$

By theorem 1, the time t^* at which the equilibrium path first reaches the steady state under forward looking management and constant real interest rates is given by

$$t^* = \chi^{-1} \left| \bar{K} - K(0) \right| \tag{26}$$

Since t^* converges to zero as the capital adjustment rate χ approaches infinity, the equilibrium path instantaneously coincides with the steady state in the limit as the capital adjustment rate becomes large. Theorem 2 says that the equilibrium path may overshoot the steady state if managers are sufficiently myopic and real interest rates are constant.

Theorem 2. *If β is sufficiently small and $\varphi = \overline{\varphi}$ then there exists $\eta > 0$ and $h \in H^*$ such that $K(0) \neq \bar{K}$ and for all $t \leq \chi^{-1} \left|\bar{K} - K(0)\right|$*

$$K(t) = K(0) + \chi \operatorname{sgn}\left(\bar{K} - K(0)\right) \tag{27}$$

The user cost of capital depends on both the opportunity cost of capital and the rate of change in the price of capital, so changes in the price of capital can distort the user cost of capital. Accordingly, falling capital prices can incentivize myopic managers to underinvest in capital and rising capital prices can incentivize myopic managers to overinvest in capital. Since the marginal product of labor is increasing in capital, distortions in capital employment produce distortions in labor employment and output, so myopic management can amplify the effect of effect of unanticipated shocks and produce endogenous deviations from the steady state. Let ψ_A denote a monetary policy under which the real interest rate varies depending on total output, total capital employment, and capital prices.

$$\varphi_A\left(h^t\right) = \frac{GK(t)^{-\gamma} + \operatorname{sgn}\left(Y(t) - \bar{Y}\right)\left[\kappa\chi QK(t)^{\kappa-1} - 1\right]}{p(t)}$$

Theorem 3 says that the equilibrium path always converges on the steady state as quickly as possible under the monetary policy φ_A, regardless of whether managers are forward looking or myopic. Equation (30) says that the path moves toward the steady state as quickly as possible. The monetary policy φ_A incentivizes myopic mangers to adopt forward looking strategies by keeping the user cost of capital below the return on capital whenever output is below its steady state value and keeping the user cost of capital above the return on capital whenever output is above its steady state value.

Theorem 3. *If $\varphi = \varphi_A$ and $h \in H^*$ then for all $t \in \mathbb{R}_+$*

$$\lim_{t \to \infty} h(t) = \bar{h} \tag{28}$$

$$\frac{d}{dt}\left|h(t) - \bar{h}\right| < 0 \tag{29}$$

$$\dot{K}(t) = \chi\left(\bar{K} - K(t)\right) \tag{30}$$

4 Conclusion

This paper studies the relationship between economic instability and myopic management in markets where interest rates depend on monetary policy. In particular, it shows that sufficiently active monetary policy can incentivize myopic managers to adopt forward looking strategies. Forward looking managers maximize the present value of future profits over an infinite horizon while myopic managers maximize the present value of future profits over a finite horizon. If manager's are forward looking, then output, wages, prices, and employment are shown to converge on the steady state as quickly as possible following an unanticipated shock. In contrast, myopic management can lead to booms and busts that overshoot the steady state. These deviations from the steady state may be triggered by unanticipated shocks or they may emerge endogenously.

Firms own the physical capital they employ, so the user cost of capital depends partly on the rate of change in the price of physical capital. Changes in the price of physical capital can temporarily raise or lower the user cost of capital, incentivizing myopic managers to underinvest or overinvest in capital assets. Monetary policy can incentivize myopic mangers to adopt forward looking strategies by neutralizing distortions to the user cost of capital. When output is below the steady state, this policy lowers interest rates enough to keep the cost of capital below the return on capital. When output is above the steady state, it raises interest rates enough to keep the cost of capital above the return on capital. These results suggest that sufficiently active monetary policy can counteract the economic instability produced by myopic management. Future research should test these results empirically.

A Proofs

Lemma 1. *There exists $G_L, G_Y > 0$ and $\gamma_L, \gamma_Y \in (0,1)$ such that*

$$L(t) = G_L K(t)^{\gamma_L} \qquad Y(t) = G_Y K(t)^{\gamma_Y} \tag{31}$$

$$\gamma_L = \frac{1-a}{1-a+b} \qquad \gamma_Y = 1 - a + \gamma_L a \tag{32}$$

Proof of proposition 1. By the first order condition on $\ell_i(t)$, the wage is equal to the marginal product of labor in equilibrium

$$w(t) = aA\ell_i(t)^{a-1} k_i(t)^{1-a}$$
$$w(t)^{\frac{1}{1-a}} \ell_i(t) = a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} k_i(t)$$

Substituting in $w(t) = BL(t)^b$ yields

$$B^{\frac{1}{1-a}} L(t)^{\frac{b}{1-a}} \ell_i(t) = a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} k_i(t)$$
$$L(t)^{\frac{b}{1-a}} \ell_i(t) = a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} B^{\frac{1}{a-1}} k_i(t)$$

Integrating both sides over $i \in [0,1]$ obtains

$$L(t)^{\frac{b}{1-a}} L(t) = a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} B^{\frac{1}{1-a}} K(t)$$
$$L(t) = \left[a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} B^{\frac{1}{a-1}}\right]^{\frac{1-a}{1-a+b}} K(t)^{\frac{1-a}{1-a+b}}$$
$$L(t) = G_L K(t)^{\gamma_L}$$

Substituting this into $Y(t) = AL(t)^a K(t)^{1-a}$ yields

$$Y(t) = AG_L^a K(t)^{\gamma_L a} K(t)^{1-a}$$
$$Y(t) = AG_L^a K(t)^{1-a+\gamma_L a}$$
$$Y(t) = G_Y K(t)^{\gamma_Y}$$

□

Lemma 2. *There exists $G_w, G_\ell, G_y > 0$ and $\gamma_w, \gamma_\ell, \gamma_y > 0$ such that*

$$w(t) = G_w K(t)^{\gamma_w} \qquad \ell_i(t) = G_\ell K(t)^{-\gamma_\ell} k_i(t) \qquad y_i(t) = G_y K(t)^{-\gamma_y} k_i(t)$$
$$G_w = BG_L^b \qquad G_\ell = a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} G_w^{-\frac{1}{1-a}} \qquad G_y = AG_\ell^a$$
$$\gamma_w = \frac{b(1-a)}{1-a+b} \qquad \gamma_\ell = \frac{b}{1-a+b} \qquad \gamma_y = \frac{ab}{1-a+b}$$

Proof. By proposition 1 we have

$$L(t) = G_L K(t)^{\frac{1-a}{1-a+b}}$$
$$BL(t)^b = BG_L^b K(t)^{\frac{b(1-a)}{1-a+b}}$$
$$w(t) = G_w K(t)^{\gamma_w}$$

Since the wage is equal to the marginal product of labor we have

$$w(t) = aA\ell_i(t)^{a-1} k_i(t)^{1-a}$$
$$G_w K(t)^{\frac{b(1-a)}{1-a+b}} = aA\ell_i(t)^{a-1} k_i(t)^{1-a}$$
$$G_w K(t)^{\frac{b(1-a)}{1-a+b}} \ell_i(t)^{1-a} = aAk_i(t)^{1-a}$$
$$G_w^{\frac{1}{1-a}} K(t)^{\frac{b}{1-a+b}} \ell_i(t) = a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} k_i(t)^{1-a}$$
$$\ell_i(t) = a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} G_w^{-\frac{1}{1-a}} K(t)^{-\frac{b}{1-a+b}} k_i(t)$$
$$\ell_i(t) = G_\ell K(t)^{-\gamma_\ell} k_i(t)$$

Substituting this into the production function yields

$$y_i(t) = A\left[G_\ell K(t)^{-\frac{b}{1-a+b}} k_i(t)\right]^a k_i(t)^{1-a}$$
$$y_i(t) = AG_\ell^a K(t)^{-\frac{ab}{1-a+b}} k_i(t)^a k_i(t)^{1-a}$$
$$y_i(t) = AG_\ell^a K(t)^{-\frac{ab}{1-a+b}} k_i(t)$$
$$y_i(t) = G_y K(t)^{-\gamma_y} k_i(t)$$

□

Proof of proposition 2. By lemma 2 we have

$$w(t)\ell_i(t) = \left[G_w K(t)^{\frac{b(1-a)}{1-a+b}}\right]\left[G_\ell K(t)^{-\frac{b}{1-a+b}} k_i(t)\right]$$
$$w(t)\ell_i(t) = G_w G_\ell K(t)^{\frac{b-ab}{1-a+b}} K(t)^{-\frac{b}{1-a+b}} k_i(t)$$
$$w(t)\ell_i(t) = G_w G_\ell K(t)^{-\frac{ab}{1-a+b}} k_i(t)$$
$$y_i(t) - w(t)\ell_i(t) = y_i(t) - G_w G_\ell K(t)^{-\frac{ab}{1-a+b}} k_i(t)$$
$$y_i(t) - w(t)\ell_i(t) = G_y K(t)^{-\frac{ab}{1-a+b}} k_i(t) - G_w G_\ell K(t)^{-\frac{ab}{1-a+b}} k_i(t)$$
$$y_i(t) - w(t)\ell_i(t) = [G_y - G_w G_\ell] K(t)^{-\frac{ab}{1-a+b}} k_i(t)$$
$$y_i(t) - w(t)\ell_i(t) = GK(t)^{-\gamma} k_i(t)$$

Here $G > 0$ since $a \in (0,1)$ and by lemma 2

$$\begin{aligned} G &= G_y - G_w G_\ell \\ &= AG_\ell^a - G_w G_\ell \\ &= A\left[a^{\frac{a}{1-a}} A^{\frac{a}{1-a}} G_w^{-\frac{a}{1-a}}\right] - G_w\left[a^{\frac{1}{1-a}} A^{\frac{1}{1-a}} G_w^{-\frac{1}{1-a}}\right] \\ &= A^{\frac{1}{1-a}} a^{\frac{a}{1-a}} G_w^{-\frac{a}{1-a}} - A^{\frac{1}{1-a}} a^{\frac{1}{1-a}} G_w^{-\frac{a}{1-a}} \\ &= A^{\frac{1}{1-a}}\left[a^{\frac{a}{1-a}} - a^{\frac{1}{1-a}}\right] G_w^{-\frac{a}{1-a}} > 0 \end{aligned}$$

Now by the definition of $\pi_i(t)$ we have

$$\pi_i(t) = y_i(t) - w(t)\ell_i(t) - c(t)k_i(t)$$
$$\pi_i(t) = GK(t)^{-\gamma} k_i(t) - c(t)k_i(t)$$
$$\pi_i(t) = \left[GK(t)^{-\gamma} - c(t)\right] k_i(t)$$
$$\pi_i(t) = \left[GK(t)^{-\gamma} + \dot{p}(t) - r(t)p(t)\right] k_i(t)$$
$$\pi_i(t) = \alpha(t)k_i(t)$$

□

Definition 1. $\lambda\left(\sigma^t, h^\tau\right) = \displaystyle\int_\tau^{t+\beta} R(s,t)\,\alpha(s)\,ds$

Lemma 3. *$\sigma^t \in \Sigma^* (h^t)$ if and only if for all $\tau \in [t, t+\beta)$*

$$\sigma_i^t (h^\tau) = \begin{cases} \chi & \text{if } \lambda (\sigma^t, h^\tau) > 0 \\ -\chi & \text{if } \lambda (\sigma^t, h^\tau) < 0 \end{cases}$$

Proof. By proposition 2 we have

$$\begin{aligned} \int_t^{t+\beta} R(s,t)\,\pi_i(s)\,ds &= \int_t^{t+\beta} R(s,t)\,\alpha(t)\,k_i(t)\,ds \\ &= \int_t^{t+\beta} ds\,R(s,t)\,\alpha(s)\left[k_i(t) + \int_t^s d\tau\,\sigma_i^t(h^\tau)\right] \\ &= C_0(h, k_i(t)) + \int_t^{t+\beta} ds\,R(s,t)\,\alpha(s)\int_t^s d\tau\,\dot{k}_i(\tau) \\ &= C_0(h, k_i(t)) + \int_t^{t+\beta} d\tau\,\dot{k}_i(\tau)\int_\tau^{t+\beta} ds\,R(s,t)\,\alpha(s) \end{aligned}$$

If $\dot{k}_j(\tau) = \sigma_j^t(h^\tau)$ for $j \in [0,1]$ then

$$\int_t^{t+\beta} R(s,t)\,\pi_i(s)\,ds = C_0(h, k_i(t)) + \int_t^{t+\beta} d\tau\,\sigma_i^t(h^\tau)\,\lambda\left(\sigma^t, h^\tau\right)$$

Now since $\sigma_i^t(h^\tau) = x_i(\tau)$ is right continuous in τ, the strategy σ_i^t satisfies the equilibrium condition (20) if and only if for all $\tau \in [t, t+\beta)$

$$\sigma_i^t (h^\tau) = \begin{cases} \chi & \text{if } \lambda (\sigma^t, h^\tau) > 0 \\ -\chi & \text{if } \lambda (\sigma^t, h^\tau) < 0 \end{cases}$$

□

Lemma 4. *If $\varphi = \overline{\varphi}$ then there exists $h \in H^*$ such that*

$$\dot{K}(t) = \chi \operatorname{sgn}\left(\bar{K} - K(t)\right)$$

Proof. For $t \in \mathbb{R}_+$ let $\sigma^t \in \Sigma$ such that $\sigma_i^t(h^t) = \chi \operatorname{sgn}\left(\bar{K} - K(t)\right)$. If $x_i(t) = \sigma_i^t(h^t)$ for all $i \in [0,1]$ and all $t \in \mathbb{R}_+$ then $K(t)$ is given by

$$K(t) = \begin{cases} \chi \operatorname{sgn}\left(\bar{K} - K(0)\right) & \text{if } t < t^* \\ 0 & \text{if } t \geq t^* \end{cases}$$

$$t^* = \chi^{-1}\left|\bar{K} - K(0)\right|$$

Hence $\dot{K}(t) = \chi \operatorname{sgn}\left(\bar{K} - K(t)\right)$. If $\varphi = \overline{\varphi}$ then by proposition 2

$$\alpha(t) = GK(t)^{-\gamma} + \dot{p}(t) - r(t)\,p(t)$$

$$\alpha(t) = GK(t)^{-\gamma} + \kappa Q \dot{K}(t)\,K(t)^{\kappa-1} - \bar{r} Q K(t)^\kappa$$

Now by proposition 3 we have

$$\alpha(t) > 0 \text{ if } \bar{K} > K(t)$$

$$\alpha(t) = 0 \text{ if } \bar{K} = K(t)$$

$$\alpha(t) < 0 \text{ if } \bar{K} > K(t)$$

Then by the definition of $\lambda(\sigma^t, h^\tau)$ we have

$$\lambda\left(\sigma^t, h^\tau\right) > 0 \text{ if } \bar{K} > K(\tau)$$
$$\lambda\left(\sigma^t, h^\tau\right) = 0 \text{ if } \bar{K} = K(\tau)$$
$$\lambda\left(\sigma^t, h^\tau\right) < 0 \text{ if } \bar{K} > K(\tau)$$

By proposition 3 $\sigma^t \in \Sigma^*(h^t)$ for all $t \in \mathbb{R}_+$, so $h \in H^*$. □

Proof of proposition 3. If $\bar{h}$ is a steady state then there exists $h \in H^*$ and $\tau > 0$ such that for all $t > \tau$ we have $h(t) = \bar{h}$. Then by proposition 2, for all $t > \tau$ we have $\alpha(t) = G\bar{K}^{-\gamma} - \bar{r}Q\bar{K}^{\kappa}$. By the definition of $\lambda(\sigma^t, h^\tau)$, we have

$$\begin{aligned}
\lambda\left(\sigma^t, h^t\right) &= \int_t^{t+\beta} R(s,t)\,\alpha(s)\,ds \\
&= \int_t^{t+\beta} e^{\bar{r}(s-t)}\left[G\bar{K}^{-\gamma} - \bar{r}Q\bar{K}^{\kappa}\right] ds \\
&= \left[G\bar{K}^{-\gamma} - \bar{r}Q\bar{K}^{\kappa}\right] \int_t^{t+\beta} e^{\bar{r}(s-t)} ds
\end{aligned}$$

By lemma (3), we have $\lambda(\sigma^t, h^t) = 0$ for $t > \tau$ so

$$\bar{K} = \left(\frac{G}{\bar{r}Q}\right)^{\frac{1}{\kappa+\gamma}}$$

□

Lemma 5. $e^{\bar{r}t} \int_t^\infty d\tau\, e^{-\bar{r}\tau} \dot{p}(\tau) = e^{\bar{r}t} \int_t^\infty ds\, e^{-\bar{r}s} \bar{r}\left[p(s) - p(t)\right]$

Proof. Since $p(s) = p(t) + \int_t^s d\tau\, \dot{p}(\tau)$ we have

$$\begin{aligned}
\bar{r}e^{\bar{r}t} \int_t^\infty ds\, e^{-\bar{r}s} p(s) &= \bar{r}e^{\bar{r}t} \int_t^\infty ds\, e^{-\bar{r}s}\left[p(t) + \int_t^s d\tau\, \dot{p}(\tau)\right] \\
\bar{r}e^{\bar{r}t} \int_t^\infty ds\, e^{-\bar{r}s}\left[p(s) - p(t)\right] &= \bar{r}e^{\bar{r}t} \int_t^\infty ds\, e^{-\bar{r}s} \int_t^s d\tau\, \dot{p}(\tau) \\
e^{\bar{r}t} \int_t^\infty ds\, e^{-\bar{r}s} \bar{r}\left[p(s) - p(t)\right] &= \bar{r}e^{\bar{r}t} \int_t^\infty d\tau\, \dot{p}(\tau) \int_\tau^\infty ds\, e^{-\bar{r}s} \\
e^{\bar{r}t} \int_t^\infty ds\, e^{-\bar{r}s} \bar{r}\left[p(s) - p(t)\right] &= \bar{r}e^{\bar{r}t} \int_t^\infty d\tau\, \dot{p}(\tau)\left[\frac{e^{-\bar{r}\tau}}{\bar{r}}\right] \\
e^{\bar{r}t} \int_t^\infty ds\, e^{-\bar{r}s} \bar{r}\left[p(s) - p(t)\right] &= e^{\bar{r}t} \int_t^\infty d\tau\, e^{-\bar{r}\tau} \dot{p}(\tau)
\end{aligned}$$

□

Definition 2. $z(t) = \int_0^\infty ds\, e^{-\bar{r}s} GK(t+s)^{-\gamma}$

Lemma 6. *If $\varphi = \overline{\varphi}$ and $\beta = \infty$ then $\lambda(\sigma^t, h^t) = z(t) - p(t)$*

Proof. By the definition of $\lambda(\sigma^t, h^t)$, if $\varphi = \overline{\varphi}$ and $\beta = \infty$ then

$$\begin{aligned}\lambda\left(\sigma^t, h^t\right) &= \int_t^\infty ds\, e^{\bar{r}(t-s)}\alpha(s) \\ &= e^{\bar{r}t}\int_t^\infty ds\, e^{-\bar{r}s}\left(GK(s)^{-\gamma} + \dot{p}(s) - \bar{r}p(s)\right)\end{aligned}$$

Then by lemma 5 we have

$$\begin{aligned}\lambda\left(\sigma^t, h^t\right) &= e^{\bar{r}t}\int_t^\infty ds\, e^{-\bar{r}s}\left(GK(s)^{-\gamma} + \bar{r}\left[p(s) - p(t)\right] - \bar{r}p(s)\right) \\ &= e^{\bar{r}t}\int_t^\infty ds\, e^{-\bar{r}s}\left(GK(s)^{-\gamma} - \bar{r}p(t)\right) \\ &= z(t) - p(t)\end{aligned}$$

□

Proof of theorem 1. If $\beta = \infty$ and $\varphi = \overline{\varphi}$ then by lemma 6

$$\begin{aligned}\lambda\left(\sigma^t, h^t\right) &= z(t) - p(t) \\ &= \int_0^\infty ds\, e^{-\bar{r}s}GK(t+s)^{-\gamma} - QK(t)^\kappa \\ &\leq \int_0^\infty ds\, e^{-\bar{r}s}G\left[K(t) + \chi s\right]^{-\gamma} - QK(t)^\kappa\end{aligned}$$

Taking the limit of the right hand side as $K(t)$ approaches zero obtains

$$\int_0^\infty ds\, e^{-\bar{r}s}G\left[\chi s\right]^{-\gamma} > 0$$

Then by lemma 3 there exists $A_0 > 0$ such that $\dot{K}(t) = \chi$ if $K(t) < A_0$ for all $h \in H^*$. Now if $K(t) \geq A_0$ then by lemma 6 we have

$$\begin{aligned}\lambda\left(\sigma^t, h^t\right) &= \int_0^\infty ds\, e^{-\bar{r}s}GK(t+s)^{-\gamma} - QK(t)^\kappa \\ &\geq \int_0^\infty ds\, e^{-\bar{r}s}GA_0^{-\gamma} - QK(t)^\kappa \\ &= \bar{r}^{-1}GA_0^{-\gamma} - QK(t)^\kappa\end{aligned}$$

Taking the limit of the right hand side as $K(t)$ approaches infinity obtains

$$\lim_{K(t)\to\infty}\left(\bar{r}^{-1}GA_0^{-\gamma} - QK(t)^\kappa\right) = \infty$$

Then by lemma 3 there exists $B_0 > 0$ such that $\dot{K}(t) = -\chi$ if $K(t) > B_0$ for all $h \in H^*$. By lemma 4 we know that $A_0 \leq \bar{K} \leq B_0$. Hence there exists $A_* \leq \bar{K} \leq B_*$ such that

$$B_* = \sup\left\{K(t) : \dot{K}(t) > -\chi, h \in H^*\right\}$$

$$A_* = \inf\left\{K(t) : \dot{K}(t) < \chi, h \in H^*\right\}$$

By lemma 3, if $K(t) \geq A_*$ then there exists $h \in H^*$ such that

$$0 \geq \lambda\left(\sigma^t, h^t\right) \geq \bar{r}^{-1}GB_*^{-\gamma} - QB_*^\kappa$$

and if $K(t) \geq A_*$ then here exists $h \in H^*$ such that

$$0 \leq \lambda\left(\sigma^t, h^t\right) \leq \bar{r}^{-1} G A_*^{-\gamma} - Q A_*^{\kappa}$$

Now since $A_* \leq B_*$ we have

$$\bar{r}^{-1} G A_*^{-\gamma} - Q A_*^{\kappa} = 0 = \bar{r}^{-1} G B_*^{-\gamma} - Q B_*^{\kappa}$$

$$A_* = B_* = \left(\frac{G}{rQ}\right)^{\frac{1}{\kappa+\gamma}} = \bar{K}$$

Then by lemma 3 for all $h \in H^*$

$$\dot{K}(t) = \chi \operatorname{sgn}\left(\bar{K} - K(t)\right)$$

Hence $K(t)$ converges to $\bar{K}$ as quickly as possible so $L(t)$ and $Y(t)$ converge to $\bar{L}$ and $\bar{Y}$ as quickly as possible by lemma 2. $\square$

Proof of theorem 2. Let $\varphi = \overline{\varphi}$. If $K(t) = \bar{K} + \varepsilon$ and $\hat{\sigma}_i^{\tau}(h^t) = \chi$ for $t \in [\tau, \tau + \beta)$ then

$$\lambda\left(\hat{\sigma}_i^{\tau}, h^t\right) = \int_t^{t+\beta} R(s,t)\,\alpha(s)\,ds$$

$$\beta^{-1}\lambda\left(\hat{\sigma}_i^{\tau}, h^t\right) = e^{rt}\beta^{-1}\int_t^{t+\beta} e^{-rs}\left[GK(t)^{-\gamma} + \dot{p}(t) - r(t)\,p(t)\right] ds$$

$$\beta^{-1}\lambda\left(\hat{\sigma}_i^{\tau}, h^t\right) = e^{rt}\beta^{-1}\int_t^{t+\beta} e^{-rs}\left[GK(t)^{-\gamma} + \kappa\chi Q K(t)^{\kappa-1} - r(t)\,QK(t)^{\kappa}\right] ds$$

Taking the limit as $\beta \to 0$ obtains

$$\lim_{\beta\to 0}\beta^{-1}\lambda\left(\hat{\sigma}_i^{\tau}, h^t\right) = GK(t)^{-\gamma} + \kappa\chi Q K(t)^{\kappa-1} - r(t)\,QK(t)^{\kappa}$$

Taking the limit as $\varepsilon \to 0$ obtains

$$\lim_{\varepsilon\to 0}\lim_{\beta\to 0}\beta^{-1}\lambda\left(\hat{\sigma}_i^{\tau}, h^t\right) = G\bar{K}^{-\gamma} + \kappa\chi Q\bar{K}^{\kappa-1} - \bar{r}Q\bar{K}^{\kappa}$$

Now since $\bar{K} = \left(\frac{G}{Q\bar{r}}\right)^{\frac{1}{\kappa+\gamma}} > 0$ we have

$$\lim_{\varepsilon\to 0}\lim_{\beta\to 0}\beta^{-1}\lambda\left(\hat{\sigma}_i^{\tau}, h^t\right) > G\bar{K}^{-\gamma} - \bar{r}Q\bar{K}^{\kappa} = 0$$

Hence there exists $c_1, c_2 > 0$ such that $\lambda(\sigma^{\tau}, h^t) > 0$ if $\varepsilon < c_1$and $\beta < c_2$. For $t \in \mathbb{R}_+$ let $\sigma^t \in \Sigma$ such that

$$\sigma_i^t(h^s) = \begin{cases} \chi \operatorname{sgn}\left(\bar{K} - K(t)\right) & \text{if } t < t_1 \\ -\chi \operatorname{sgn}\left(\bar{K} - K(t)\right) & \text{if } t > t_1 \text{ and } s \leq t_2 \\ 0 & \text{if } t > t_1 \text{ and } s > t_2 \end{cases}$$

$$t_1 = \chi^{-1}\left|\bar{K} - K(0)\right| + \chi^{-1}\varepsilon$$

$$t_2 = t_1 + \chi^{-1}\varepsilon$$

Then $\sigma^t \in \Sigma^* \left(h^t\right)$ by lemma 3, so there exists $h \in H^*$ such that $K(0) < \bar{K}$ and

$$K(t) = \begin{cases} K(0) + \chi t & \text{if } t \leq t_1 \\ K(0) + 2\chi t_1 - \chi t & \text{if } t_1 < t < t_2 \\ \bar{K} & \text{if } t \geq t_2 \end{cases}$$

□

Proof of theorem 3. By definition, if $\varphi = \varphi_A$ then

$$r(t) = \varphi_A\left(h^t\right) = \frac{GK(t)^{-\gamma} + \operatorname{sgn}\left(Y(t) - \bar{Y}\right)\left[\kappa\chi QK(t)^{\kappa-1} - 1\right]}{p(t)}$$

$$r(t)\,p(t) = GK(t)^{-\gamma} + \operatorname{sgn}\left(Y(t) - \bar{Y}\right)\left[\kappa\chi QK(t)^{\kappa-1} - 1\right]$$

By proposition 3 and lemma 2 we have $\operatorname{sgn}\left(Y(t) - \bar{Y}\right) = \operatorname{sgn}\left(K(t) - \bar{K}\right)$ and

$$r(t)\,p(t) = GK(t)^{-\gamma} + \chi \operatorname{sgn}\left(K(t) - \bar{K}\right)\kappa QK(t)^{\kappa-1} - \operatorname{sgn}\left(K(t) - \bar{K}\right)$$

$$\operatorname{sgn}\left(K(t) - \bar{K}\right) = GK(t)^{-\gamma} + \chi \operatorname{sgn}\left(K(t) - \bar{K}\right)\kappa QK(t)^{\kappa-1} - r(t)\,p(t)$$

By proposition 2, since $\dot{K}(t) \in [-\chi, \chi]$ we have

$$\alpha(t) > 0 \quad \text{if } \bar{K} > K(t) \tag{33}$$

$$\alpha(t) = 0 \quad \text{if } \bar{K} = K(t) \tag{34}$$

$$\alpha(t) < 0 \quad \text{if } \bar{K} < K(t) \tag{35}$$

Suppose for contradiction that there exists $\tau \in \mathbb{R}_+$ such that $\bar{K} < K(\tau)$ and $\dot{K}(\tau) \neq -\chi$. Since $\bar{K} < K(\tau)$ we have $\alpha(\tau) < 0$ by equation (35). Since $\dot{K}(\tau) \neq -\chi$ we have $\lambda(\sigma^\tau, h^\tau) \geq 0$ by lemma 3. By the definition of $\lambda(\sigma^\tau, h^\tau)$ differentiating with respect to τ obtains $\dot{\lambda}(\sigma^\tau, h^\tau) = -R(\tau, t)\,\alpha(\tau) > 0$ since $\alpha(\tau) < 0$. Hence for all $t > \tau$ we have $\dot{\lambda}(\sigma^\tau, h^\tau) > 0$ and $\dot{K}(t) > 0$ so $K(t) > K(\tau) > \bar{K}$ and $\alpha(t) < 0$. But then by definition the definition of $\lambda(\sigma^\tau, h^\tau)$ we have

$$\lambda(\sigma^\tau, h^\tau) = \int_\tau^{\tau+\beta} R(s,t)\,\alpha(s)\,ds < 0$$

So we must have $\dot{K}(\tau) = -\chi$ if $K(t) > \bar{K}$. By the same argument, we must have $\dot{K}(\tau) = \chi$ if $K(t) < \bar{K}$. Hence for all $h \in H^*$

$$K(t) = \begin{cases} \chi \operatorname{sgn}\left(\bar{K} - K(0)\right) & \text{if } t < t^* \\ 0 & \text{if } t \geq t^* \end{cases}$$

$$t^* = \chi^{-1}\left|\bar{K} - K(0)\right|$$

Thus $K(t)$ converges to $\bar{K}$ as quickly as possible, so $L(t)$ converges to $\bar{L}$ as quickly as possible by 2. Hence $h(t)$ converges to $\bar{h}$ as quickly as possible. □

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