Knowledge, Interest Rates, and Asset Price Bubbles

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Abstract

This paper characterizes the relationship between interest rates, investment horizons, and the size of price bubbles under finite order knowledge of rationality. An outcome is said to be rationalizable if it is consistent with the assumption that all agents possess common knowledge of rationality. As the transaction rate approaches infinity, the unique rationalizable price path is shown to instantaneously coincide with the fundamental value. Under finite transaction rates, the unique rationalizable price path is shown to converge on the fundamental value as quickly as possible. In contrast, rational agents with finite order knowledge of rationality are shown to generate price bubbles that deviate from the rationalizable price path. Lower interest rates and shorter investment horizons are shown to produce larger bubbles under every finite order knowledge of rationality.

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1 Introduction

An outcome is said to be rationalizable[1] if it is consistent with the assumption that agents possess common knowledge[2] of rationality. A rational investor who is uncertain about the rationality of others may expect non-rationalizable prices, leading them to purchase assets at prices that exceed fundamentals. If rational investors hold such beliefs, realized prices may indeed be non-rationalizable. Hence rational agents with first order knowledge of rationality may similarly expect non-rationalizable prices. Continuing this way, finite order knowledge of rationality is never sufficient for rationalizability. Such deviations from the rationalizable price path are self-fulfilling prophecies in that they occur because investors anticipate non-rationalizable prices.

This paper characterizes the relationship between interest rates, investment horizons, and the size of price bubbles under finite order knowledge of rationality in continuous time asset markets. We begin by characterizing the unique rationalizable price path under common knowledge of rationality. As transaction rates approach infinity, the rationalizable price path instantaneously coincides with the fundamental value. Under finite transaction rates, rationalizable prices converge on the fundamental value as quickly as possible. Conversely, a bubble is said to occur when prices move away from the fundamental value. Common knowledge of rationality is sufficient for the elimination of bubbles, but finite order knowledge of rationality is always insufficient for the elimination of bubbles. For all $n \in \mathbb{N}$, we demonstrate the existence of price bubbles consistent with n^{th} order knowledge of rationality.

We show that price bubble amplitude is strictly decreasing in the interest rate and the expected investment horizon under every finite order knowledge of rationality. Lower interest rates and shorter investment horizons produce

[1] This notion of rationalizability is discussed by Bernheim (1984) and Pearce (1984).

[2] A rational agent who knows that all other agents are rational is said to have first order knowledge of rationality. A rational agent who knows that all other agents have first order knowledge of rationality is said to have second order knowledge of rationality. Proceeding this way, rational agents who possess every finite order knowledge of rationality are said to have common knowledge of rationality (Aumann, 1995).

larger price bubbles. Intuitively, this occurs because rational investors evaluate assets at a convex combination between the fundamental value and expected future prices. In the limit as the interest rate or the expected investment horizon becomes large, asset valuations coincide with the fundamental value. Shorter investment horizons and lower interest rates cause rational investors to place less weight on the fundamental value and more weight on subjective beliefs about future prices.

These results contrast with those of Allen, Morris, and Postlewaite (1993) and Conlon (2004) who describe equilibrium overpricing in markets with asymmetric information about dividends. In contrast, the present paper describes rational deviations from equilibrium in markets with symmetric information about dividends. Farhi and Tirole (2012), Martin and Ventura (2012), and Gali (2014) describes how equilibrium prices can exceed fundamental values in overlapping generations models. In contrast, the present paper describes how shorter investment horizons and lower interest rates produce larger bubbles in settings where agents possess finite order knowledge of rationality.

The remainder of the paper is organized as follows. Section 2 describes the model, section 3 presents the results, section 4 provides an example, and section 5 concludes. Proofs are provided in the appendix.

2 The Model

Consider an asset market populated by a continuum of investors indexed by $i \in [0,1]$ and a continuum of liquidity providers indexed by $\ell \in \mathbb{R}$. Let $p(t) \in \mathbb{R}_+$ denote the price of an asset at time $t \in \mathbb{R}_+$. Let $a_i(t) \in \mathbb{R}$ denote the quantity of the asset held by investor i at time t. Let $A(t)$ denote the total quantity demanded by investors at time t.

$$A(t) = \int_0^1 a_i(t)\, di \tag{1}$$

If $A(t) \geq \ell$, liquidity provider ℓ offers to sell at price $f(\ell) \in \mathbb{R}$. If $A(t) < \ell$, liquidity provider ℓ offers to buy at price $f(\ell) \in \mathbb{R}$. Liquidity providers are ordered by price such that $f(x) \leq f(y)$ for $x \leq y$. The inverse supply function $f : \mathbb{R} \to \mathbb{R}$ is assumed to be a smooth increasing bijection of polynomial order. The market clearing condition (2) requires that the total quantity demanded is equal to the total quantity supplied at each point in time.

$$p(t) = f(A(t)) \tag{2}$$

2.1 Investment

Let P denote the set of continuous and right-differentiable price paths $p : \mathbb{R}_+ \to \mathbb{R}$. Let $p_t : [0, t] \to \mathbb{R}$ denote the restriction of the price path $p \in P$ to the closed interval $[0, t]$. Let $\mathcal{H}$ denote the set of all possible price histories.

$$\mathcal{H} = \{p_t : p \in P, t \in \mathbb{R}_+\} \tag{3}$$

An investment strategy $\phi_i : \mathcal{H} \to [-\gamma, \gamma]$ specifies investor i's purchase rate at time t as a function of the price history $p_t \in \mathcal{H}$. The transaction rate $\gamma \in \mathbb{R}_{++}$ can be arbitrarily large. Let $a_i(t|\phi_i, p)$ denote the quantity of assets held by investor i at time t under the price path p and the investment strategy ϕ_i.

$$a_i(t|\phi_i, p) = a_i(0) + \int_0^t \phi_i(p_s)\, ds \tag{4}$$

Let Φ_i denote the set of all strategies $\phi_i : \mathcal{H} \to [-\gamma, \gamma]$ such that $\dot{a}_i(t|\phi_i, p)$ is right-continuous in t for all $p \in P$. Let $\Phi = \prod_{i \in [0,1]} \Phi_i$ denote the strategy space. Let $A(t|\phi, p)$ denote the total quantity of assets demanded by investors at time t under the strategy profile $\phi \in \Phi$ and the price path $p \in P$.

$$A(t|\phi, p) = \int_0^1 a_i(t|\phi_i, p)\, di \tag{5}$$

A price path p is said to be feasible if it satisfies the market clearing condition (6) for some strategy profile $\phi \in \Phi$. Let P_f denote the set of all feasible price paths.

$$p(t) = f(A(t|\phi, p)) \tag{6}$$

2.2 Wealth

Let $m_i(t) \in \mathbb{R}$ denote the quantity of money held by investor i at time t. Investor i's wealth $w_i(t)$ is equal to the value of their money plus the value of their assets.

$$w_i(t) = m_i(t) + p(t)\, a_i(t) \tag{7}$$

Let $r \in \mathbb{R}_{++}$ denote the interest rate and let $z \in \mathbb{R}_{++}$ denote the dividend rate[3] on the asset. Let $\dot{m}_i(t)$ denote the right-derivative[4] of m_i. Investor i's net revenue is equal to their interest income plus their dividend income minus the cost of their asset purchases.

$$\dot{m}_i(t) = rm_i(t) + za_i(t) - p(t)\, \dot{a}_i(t) \tag{8}$$

Let $\bar{p}$ denote the present value of all future cash flows generated by an asset.

$$\bar{p} = \int_0^\infty e^{-rt} z\, dt = \frac{z}{r} \tag{9}$$

Let T_i denote investor i's stochastic investment horizon such that $\Pr(T_i \leq t) = 1 - e^{-t/\beta}$. Hence the expected investment horizon is $E\{T_i\} = \beta$. At time T_i, investor i's portfolio is transferred to a new investor with an identically distributed investment horizon. Agent i's payoff Π_i is the the present value of their final wealth.

$$\Pi_i = e^{-rT_i} w_i(T_i) \tag{10}$$

[3]For stocks, the dividend rate corresponds to the rate at which dividends accrue to the stockholder. For real estate, the dividend rate corresponds to the rate at which rental revenue accrues to the property owner.

[4]This paper uses $\dot{x}(t)$ to denote the right-derivative of x with respect to t.

2.3 Rationality

Let Σ denote the σ-algebra on P generated[5] by $\mathcal{H}$. Let $\mu_i : \Sigma \to \mathbb{R}$ be a probability measure on P such that $\mu_i(P_f) = 1$. The probability measure μ_i describes investor i's beliefs about future prices. Let $\mu_i(B)$ denote investor i's probability that $p \in B$. Let $\mu_i(B \mid p_t)$ denote investor i's conditional probability that $p \in B$ given the price history $p_t \in \mathcal{H}$. Let M_i denote the set of all such beliefs μ_i. Let $M = \prod_{i\in[0,1]} M_i$ denote the set of all possible belief profiles. Let ϕ_{it}^{+} denote the set of strategies that agree with ϕ_i up to time t.

$$\phi_{it}^{+} = \{\varphi_i \in \Phi_i : \varphi_i(p_s) = \phi(p_s) \text{ for } p \in P \text{ and } s \leq t\} \tag{11}$$

An investor is said to be rational if they always maximize their expected payoff. A strategy ϕ_i is said to be rational under the belief μ_i if it maximizes investor i's expected payoff conditional on every possible price history. More formally, ϕ_i is rational under μ_i if

$$\phi_i \in \underset{\varphi_i \in \phi_{it}^{+}}{\operatorname{argmax}} \int E\{\Pi_i|\varphi_i, p\}\, d\mu_i(p|p_t) \qquad \text{for all } p_t \in \mathcal{H} \tag{12}$$

Let $\Phi_i^*(\mu_i)$ denote the set of rational strategies under the belief μ_i. A strategy profile ϕ is said to be rational under the belief profile μ if $\phi_i \in \Phi_i^*(\mu_i)$ for all $i \in [0,1]$. Let $\Phi^*(\mu)$ denote the set of rational strategy profiles under the belief profile μ. A strategy profile ϕ is said to be rational if $\phi \in \Phi^*(\mu)$ for a belief profile μ. Let Φ_0 denote the set of rational strategy profiles. Let $P(\mu)$ denote the set of price $p \in P$ that satisfy the market clearing condition (6) for a strategy profile $\phi \in \Phi^*(\mu)$. A price path p is said to be consistent with rationality if $p \in P(\mu)$ for some belief profile μ. Let P_0 denote the set of price paths that are consistent with rationality.

[5] That is to say, Σ is the σ-algebra generated by the subsets of P consistent with each possible history $p_t \in \mathcal{H}$. More formally $\Sigma = \sigma(\{\{q \in P : q_t = p_t\} \subseteq P : p_t \in \mathcal{H}\})$.

2.4 Knowledge

An investor is said to possess first order knowledge of rationality if they know that all other investors are rational. A belief μ_i is said to be consistent with first order knowledge of rationality if $\mu_i(P_0|p_t) = 1$ for all $p \in P_0$. Let M^1 denote the set of belief profiles that are consistent with first order knowledge of rationality. A strategy profile $\phi \in \Phi$ is said to be consistent with first order knowledge of rationality if it is rational under a belief profile $\mu \in M^1$. Let Φ^1 denote the set of strategy profiles that are consistent with first order knowledge of rationality. A price path $p \in P$ is said to be consistent with first order knowledge of rationality if it satisfies the market clearing condition for a strategy profile $\phi \in \Phi^1$. Let P_1 denote the set of price paths that are consistent with first order knowledge of rationality.

An investor is said to possess n^{th} order knowledge of rationality if they know that all other investors have $(n-1)^{th}$ order knowledge of rationality. A belief μ_i is said to be consistent with n^{th} order knowledge of rationality if $\mu_i(P_{n-1}|p_t) = 1$ for all $p \in P_{n-1}$. Let M^n denote the set of belief profiles that are consistent with n^{th} order knowledge of rationality. A strategy profile $\phi \in \Phi^{n-1}$ is consistent with n^{th} order knowledge of rationality if it is rational under $\mu \in M^n$. Let Φ^n denote the set of strategy profiles that are consistent with n^{th} order knowledge of rationality. A price path $p \in P_{n-1}$ is said to be consistent with n^{th} order knowledge of rationality if it satisfies the market clearing condition for a strategy profile $\phi \in \Phi^n$. Let P_n denote the set of price paths that are consistent with n^{th} order knowledge of rationality.

A belief profile μ is said to be consistent with common knowledge of rationality if $\mu \in M^n$ for all $n \in \mathbb{N}$. Let M^∞ denote the set of belief profiles that are consistent with common knowledge of rationality. A strategy profile $\phi \in \Phi$ is said to be rationalizable if it is rational under a belief profile $\mu \in M^\infty$. Let Φ^∞ denote the set of rationalizable strategy profiles. A price path $p \in P$ is said to be rationalizable if it satisfies the market clearing condition for a rationalizable strategy profile $\phi \in \Phi^\infty$. Let P_∞ denote the set of rationalizable price paths.

3 Results

Theorem 1 characterizes the optimal investment strategy under the belief μ_i. Rational investors value assets at a convex combination between the fundamental value $\bar{p}$ and expected future prices $\tilde{p}(t)$. They purchase assets whenever their valuation exceeds the price of the asset and sell assets whenever the price of the asset exceeds their valuation. Here γ denotes the transaction rate, α denotes the weight placed on the fundamental value, δ denotes the rate at which investors discount the future, β denotes the expected investment horizon, and r denotes the interest rate.

Theorem 1. *$\phi_i \in \Phi_i$ is rational under $\mu_i \in M_i$ if and only if, conditional on each $p_s \in \mathcal{H}$, almost surely for almost all $t \geq s$*

$$\phi_i(p_t) = \begin{cases} \gamma & \text{if } p(t) < V(p_t, \mu_i) \\ -\gamma & \text{if } p(t) > V(p_t, \mu_i) \end{cases} \tag{13}$$

$$V(p_t, \mu_i) = \alpha\bar{p} + (1-\alpha)\int \tilde{p}(t)\, d\mu_i(p|p_t) \tag{14}$$

$$\tilde{p}(t) = \delta \int_0^\infty e^{-\delta s} p(t+s)\, ds \tag{15}$$

$$\alpha = \frac{r}{\delta} \qquad \delta = r + \frac{1}{\beta} \tag{16}$$

Theorem 2 identifies the unique rationalizable price path. If investors possess common knowledge of rationality, prices converge on the fundamental value as quickly as possible. As the transaction rate γ becomes large, the time t^* at which the rational price path reaches the fundamental value approaches zero. Hence rationalizable prices instantaneously coincide with the fundamental value in the limit as the transaction rate approaches infinity.

Theorem 2. *There is a unique rationalizable price path p^* such that*

$$p^*(t) = \begin{cases} f\left(A(0) + \gamma t \operatorname{sgn}(\bar{p} - p(0))\right) & \text{if } t < t^* \\ \bar{p} & \text{if } t \geq t^* \end{cases} \tag{17}$$

$$t^* = \gamma^{-1}\left|f^{-1}(\bar{p}) - A(0)\right| \tag{18}$$

The rationalizable price path converges on the fundamental value as quickly as possible, so $\dot{p}^*(t)\left[p^*(t) - \bar{p}\right] \geq 0$ for all $t \in \mathbb{R}_+$. Conversely, a price path p is said to be in a bubble when prices are moving away from the fundamental value such that $\dot{p}(t)\left[p(t) - \bar{p}\right] < 0$. Positive bubbles occur when prices increase above the fundamental value. Negative bubbles occur when prices decrease below the fundamental value. The amplitude of a bubble is defined as the maximal deviation between the fundamental value $\bar{p}$ and the asset price $p(t)$ over the course of the bubble. Let $\Lambda(\mu, r, \beta)$ denote the maximal bubble amplitude under the belief profile μ, the interest rate r, and the expected investment horizon β. If $\dot{p}(t)\left[p(t) - \bar{p}\right] \geq 0$ for all $t \in \mathbb{R}_+$ and $p \in P(\mu)$ then $\Lambda(\mu, r, \beta) = 0$. Otherwise, the bubble amplitude is given by

$$\Lambda(\mu, r, \beta) = \sup\left\{\left|p(t) - \bar{p}\right| : \dot{p}(t)\left[p(t) - \bar{p}\right], p \in P(\mu)\right\} \tag{19}$$

Theorem 3 guarantees the existence of bubbles under every finite order knowledge of rationality. It says that for all $n \in \mathbb{N}$, there exists a belief profile μ consistent with n^{th} order knowledge of rationality and a price path $p \in P(\mu)$ exhibiting a bubble of nonzero amplitude. While theorem 2 says that infinite order common knowledge of rationality is sufficient for the elimination of bubbles, theorem 3 says that finite order knowledge of rationality is never sufficient for the elimination of bubbles.

Theorem 3. *For all $n \in \mathbb{N}$ there exists $\mu \in M_n$ such that $\Lambda(\mu, r, \beta) > 0$.*

Theorem 4 characterizes the relationship between interest rates, investment horizons, and the amplitude of bubbles. It states that the maximal bubble amplitude is strictly decreasing in the interest rate and the expected investment horizon under all belief profiles that fail to eliminate bubbles. Since n^{th}

order knowledge of rationality is never sufficient for the elimination of bubbles, lower interest rates and shorter expected investment horizons produce larger bubbles under every finite order knowledge of rationality.

Theorem 4. *The maximal bubble amplitude* $\Lambda(\mu, r, \beta)$ *is strictly decreasing in both the interest rate* r *and the expected investment horizon* β *if* $\Lambda(\mu, r, \beta) > 0$.

4 Example

This section provides a simple example illustrating the results presented in section 3. If $A(0) = \bar{p}$ then the unique rationalizable price path is $p^*(t) = \bar{p}$ by theorem 2. If $f(x) = x$ then the price path $b \in P$ that perpetually increases as quickly as possible is given by $b(t) = \bar{p} + \gamma t$. Let $g : P \times \mathbb{R}_+ \to P$ such that $g(p, s) \in P$ is consistent with p until time s and converges on the fundamental value as quickly as possible thereafter.

$$g(p,s)(t) = \begin{cases} p(t) & \text{if } t \leq s \\ p(s) + (t-s)\gamma \operatorname{sgn}(\bar{p} - p(s)) & \text{if } s \leq t \leq \tau^*_{ps} \\ \bar{p} & \text{if } t \geq \tau^*_{ps} \end{cases} \tag{20}$$

$$\tau^*_{ps} = \gamma^{-1} |\bar{p} - p(s)| \tag{21}$$

4.1 Rationality

Let $\mu_i^0 \in M$ denote the beliefs held by a rational investor who initially expects the perpetual bubble b such that $\mu_i^0(b|p_t) = 1$ if $p_t = b_t$. Conditional on any deviation from b, they expect prices to converge on the fundamental value as quickly as possible so that $\mu_i^0(g(p,t)\,|\,p_t) = 1$ if $p_t \neq b_t$. By theorem 1, the rational asset valuation satisfies

$$V\left(b_t, \mu_i^0\right) = \alpha\bar{p} + (1-\alpha)\left[b(t) + \frac{\gamma}{\delta}\right]$$

Hence the price path $b(t)$ lies below the rational valuation $V(b_t, \mu_i^0)$ before

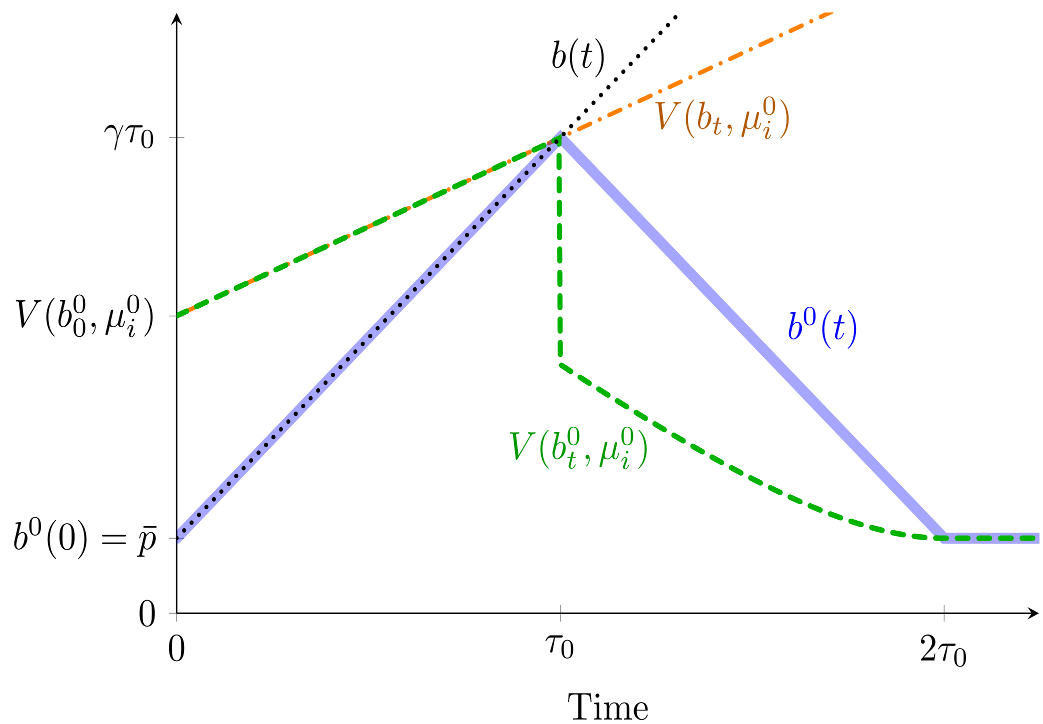

Figure 1: The dotted line depicts the initially anticipated price path $b(t)$. The dot-dashed line depicts the asset valuation $V(b_t, \mu_i^0)$ under the price path $b(t)$. The solid line depicts the price bubble $b^0(t)$ consistent with rationality. The dashed line depicts the asset valuation $V(b_t^0, \mu_i^0)$ under the price path $b^0(t)$.

time $\tau_0 = \left(\beta r^2 + r\right)^{-1}$, coincides with the rational valuation at time τ_0, and exceeds the rational valuation thereafter.

$$\begin{aligned} b\left(t\right) &< V\left(b_t, \mu_i^0\right) \quad \text{if } t < \tau_0 \\ b\left(t\right) &= V\left(b_t, \mu_i^0\right) \quad \text{if } t = \tau_0 \\ b\left(t\right) &> V\left(b_t, \mu_i^0\right) \quad \text{if } t > \tau_0 \end{aligned}$$

As r or β become large, τ_0 approaches to 0. As r becomes small, τ_0 approaches infinity. Let ϕ_i^0 denote the investment strategy that involves buying assets as quickly as possible whenever the rational asset valuation $V\left(p_t, \mu_i^0\right)$ exceeds the price $p\left(t\right)$ and selling assets as quickly as possible whenever the price $p\left(t\right)$ exceeds the rational asset valuation $V\left(p_t, \mu_i^0\right)$.

$$\phi_i^0\left(p_t\right) = \begin{cases} \gamma & \text{if } p\left(t\right) < V\left(p_t, \mu_i^0\right) \\ -\gamma & \text{if } p\left(t\right) > V\left(p_t, \mu_i^0\right) \end{cases}$$

As illustrated in figure 1, investors who employ the rational strategy ϕ_i^0 purchase assets as quickly as possible until time τ_0. After time τ_0 they start selling assets as quickly as possible, so prices begin falling at time τ_0. When prices start falling, investors revise their beliefs and prices continue decreasing as quickly as possible. At time $2\tau_0$, prices stabilize on the fundamental value. The resulting price path $b^0 = g\left(b, \tau_0\right)$ satisfies the market clearing condition for the strategy profile ϕ^0. By theorem 1, ϕ_i^0 is rational under the belief μ_i^0, so b^0 is consistent with rationality.

4.2 Knowledge

Let $\mu_i^1 \in M^1$ denote the beliefs held by a rational investor with first order knowledge of rationality who initially expects the rational price path b^0 such that $\mu_i^1\left(b^0|p_t\right) = 1$ if $p_t = b_t^0$. Conditional on any deviation from b^0, they expect prices to converge on the fundamental value as quickly as possible so that $\mu_i^1\left(g\left(p,t\right)|\,p_t\right) = 1$ if $p_t \neq b_t^0$. By theorem 1 the rational asset valuation

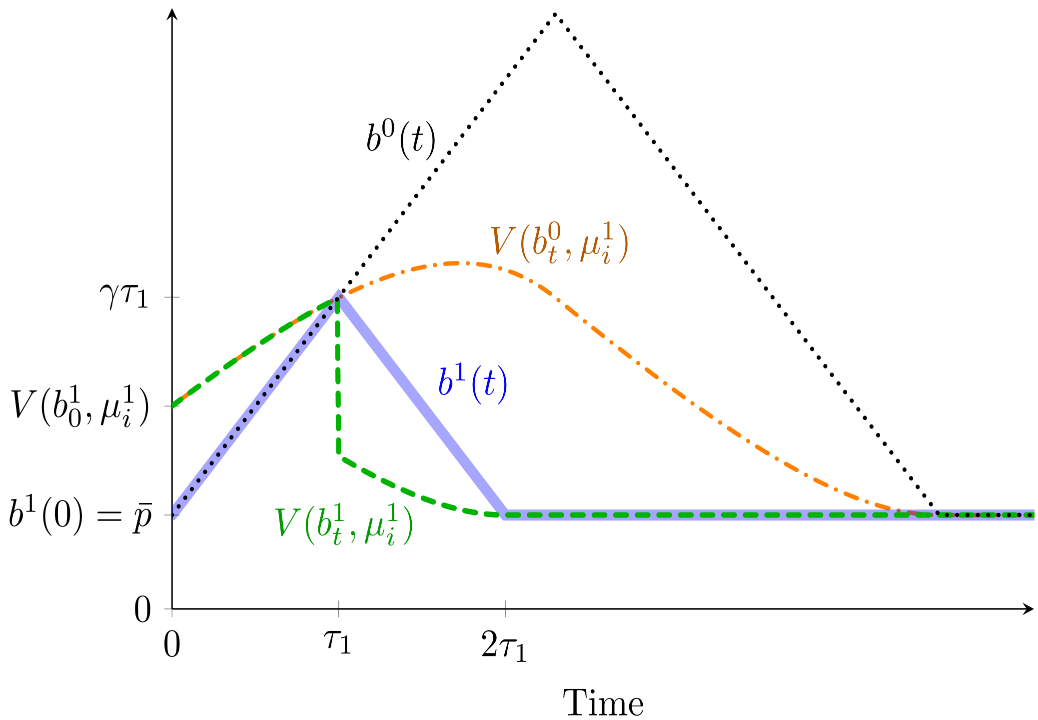

Figure 2: The dotted line depicts the initially anticipated price path $b^0(t)$ consistent with rationality. The dot-dashed line depicts the asset valuation $V(b_t^0, \mu_i^1)$ under the price path $b^0(t)$. The solid line depicts the price bubble $b^1(t)$ consistent with first order knowledge rationality. The dashed line depicts the asset valuation $V(b_t^1, \mu_i^1)$ under the price path $b^1(t)$.

satisfies

$$V\left(b_t^0,\mu_i^1\right) = \alpha\bar{p} + (1-\alpha)\,\delta\int_0^\infty e^{-\delta s} b^0\,(t+s)\,ds \tag{22}$$

$$= \bar{p} + (1-\alpha)\,\delta\int_0^\infty e^{-\delta s}\left[b^0\,(t+s) - \bar{p}\right] ds \tag{23}$$

$$= \bar{p} + (1-\alpha)\,\delta\int_{\min\{t,2\tau_0\}}^{2\tau_0} e^{-\delta(s-t)}\left[b^0\,(s) - \bar{p}\right] ds \tag{24}$$

Since b^0 is increasing over the closed interval $[0,\tau_0]$ and decreasing over the closed interval $[\tau_0, 2\tau_0]$, there exists $\tau_1 \in (0,\tau_0)$ such that

$$\begin{aligned} b^0\,(t) &< V\left(b_t^0,\mu_i^1\right) && \text{if } t<\tau_1 \\ b^0\,(t) &= V\left(b_t^0,\mu_i^1\right) && \text{if } t=\tau_1 \\ b^0\,(t) &> V\left(b_t^0,\mu_i^1\right) && \text{if } t>\tau_1 \end{aligned}$$

Let $\phi_i^1 \in \Phi_i^1$ denote the strategy that involves buying assets as quickly as possible when the rational asset valuation $V(p_t,\mu_i^1)$ exceeds the price $p\,(t)$ and selling assets as quickly as possible when the price $p\,(t)$ exceeds the rational asset valuation $V(p_t,\mu_i^1)$.

$$\phi_i^1\,(p_t) = \begin{cases} \gamma & \text{if } p\,(t) < V\,(p_t,\mu_i^1) \\ -\gamma & \text{if } p\,(t) > V\,(p_t,\mu_i^1) \end{cases}$$

As illustrated in figure 2, investors who employ the strategy ϕ_i^1 purchase assets as quickly as possible until time τ_1. Accordingly, prices increase until time τ_1, begin falling at time τ_1, and stabilize on the fundamental value at time $2\tau_1$. The resulting price path $b^1 = g\,(b,\tau_1)$ satisfies the market clearing condition for the strategy profile ϕ^1. By theorem 1, ϕ_i^1 is rational under the belief $\mu_i^1 \in M^1$, so b^1 is consistent with first order knowledge of rationality. By the same argument, there exists $\tau_2 \in (0,\tau_1)$ such that $b^2 = g\,(b,\tau_2)$ is consistent with second order knowledge of rationality. Continuing this way, for all $n \in \mathbb{N}$ there exists $\tau_n \in (0,\tau_{n-1})$ such that $b^n = g\,(b,\tau_n)$ is consistent with n^{th} order knowledge of rationality.

5 Conclusion

There is a unique rationalizable price path in continuous-time asset markets. In the limit as the transaction rate approaches infinity, rationalizable prices instantaneously coincide with fundamentals. Under finite transaction rates, rationalizable prices converge on fundamentals as quickly as possible. A rational investor who is uncertain about the rationality of others may expect deviations from the rationalizable price path. If investors hold such beliefs, realized prices may indeed diverge from the rationalizable price path. Consequently, investors with first order knowledge of rationality may expect deviations from the rationalizable price path. Continuing this way, finite order knowledge of rationality is never sufficient for the rationalizable price path.

Rational investors purchase assets as quickly as possible whenever their valuation for the asset exceeds its price, and sell assets as quickly as possible whenever the price exceeds their valuation. They value assets at a convex combinations between the fundamental value and expected future prices. Lower interest rates and shorter investment horizons cause rational investors to place less weight on fundamentals and more weight on expected future prices, leading to larger deviations from the rationalizable price path. Future research should extend this model to settings with stochastic interest rates, stochastic dividends, production, consumption, and depreciation. Additional research is needed to test these results empirically.

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A Proofs

Lemma 1. $w_i(t) = e^{rt} w_i(0) + e^{rt} \int_0^t e^{-rs} \zeta(p,s)\, a_i(s)\, ds$

$$\text{where } \zeta(p,t) = z + \dot{p}(t) - rp(t)$$

Proof. Since $\dot{m}_i(t) = rm_i(t) + za_i(t) - p(t)\dot{a}_i(t)$

$$\begin{aligned}
w_i(t) &= m_i(t) + p(t)\, a_i(t) \\
\dot{w}_i(t) &= \dot{m}_i(t) + \dot{p}(t)\, a_i(t) + p(t)\, \dot{a}_i(t) \\
&= rm_i(t) + za_i(t) + \dot{p}(t)\, a_i(t) \\
&= r\left[w_i(t) - p(t)\, a_i(t)\right] + za_i(t) + \dot{p}(t)\, a_i(t) \\
&= rw_i(t) + \left[z + \dot{p}(t) - rp(t)\right] a_i(t) \\
&= rw_i(t) + \zeta(p,t)\, a_i(t) \\
w_i(t) &= e^{rt} w_i(0) + e^{rt} \int_0^t e^{-rs} \zeta(p,s)\, a_i(s)\, ds
\end{aligned}$$

□

Lemma 2. $p(t) = O(t^c)$

Proof. Since $\phi_i(p_t) \in [-\gamma, \gamma]$

$$a_i(t|\phi, p) = a_i(0) + \int_0^t \phi_i(p_t)\, dt = O(t)$$
$$A(t|\phi, p) = \int_0^1 a_i(t|\phi, p)\, di = O(t)$$

By the market clearing condition

$$p(t) = f(A(t|\phi, p)) = f(O(t))$$
$$p(t) = O(t^c) \qquad \text{since } f(x) = O(x^c)$$

□

Lemma 3. $e^{-rt} w_i(t) = O(1)$

Proof. By lemma 2, the price path is of polynomial order

$$p(t) = f(t^c)$$
$$e^{-rt} p(t)\, a_i(t) = O(1) \quad \text{since } \dot{a}_i(t) = \phi_i(p_t) \in [-\gamma, \gamma]$$

Since $\dot{m}_i(t) = r m_i(t) + z a_i(t) - p(t)\, \dot{a}_i(t)$

$$m(t) = e^{rt} m_i(0) + e^{rt} \int_0^t e^{-rs} [z a_i(s) - p(s)\, \dot{a}_i(s)]\, ds$$
$$e^{-rt} m(t) = m_i(0) + \int_0^t e^{-rs} O(s^c)\, ds = O(1)$$
$$e^{-rt} w_i(t) = e^{-rt} [m_i(t) + p(t)\, a_i(t)] = O(1)$$

□

Lemma 4. $E\left\{\Pi_i|p,\phi_i\right\} = w_i\left(0\right) + \int_0^\infty e^{-\delta t}\zeta\left(p,t\right)a_i\left(t\right)dt$

Proof. By lemmas 1 and 3

$$
\begin{aligned}
e^{-rt}w_i\left(t\right) - w_i\left(0\right) &= \int_0^t e^{-rs}\zeta\left(p,s\right)a_i\left(s\right)ds = O\left(1\right) \\
E\left\{\Pi_i|p,\phi_i\right\} - w_i\left(0\right) &= E\left\{e^{-rT_i}w_i\left(T_i\right) - w_i\left(0\right)\right\} \\
&= E\left\{\int_0^{T_i} e^{-rs}\zeta\left(p,s\right)a_i\left(s\right)ds\right\} \\
&= \frac{1}{\beta}\int_{t=0}^{t=\infty} e^{-t/\beta}\int_{s=0}^{s=t} e^{-rs}\zeta\left(p,s\right)a_i\left(s\right)ds\,dt \\
&= \frac{1}{\beta}\int_{s=0}^{s=\infty} e^{-rs}\zeta\left(p,s\right)a_i\left(s\right)\int_{t=s}^{t=\infty} e^{-t/\beta}dt\ ds \\
&= \frac{1}{\beta}\int_{s=0}^{s=\infty} e^{-rs}\zeta\left(p,s\right)a_i\left(s\right)\left[\beta e^{-s/\beta}\right]ds \\
&= \int_0^\infty e^{-\delta s}\zeta\left(p,s\right)a_i\left(s\right)ds \qquad \text{since } \delta = r + \frac{1}{\beta}
\end{aligned}
$$

□

Lemma 5. $E\left\{\Pi_i|p,\phi_i\right\} = w_i\left(0\right) + a_i\left(0\right)\lambda\left(p,0\right) + \int_0^\infty e^{-\delta t}\phi_i\left(p_t\right)\lambda\left(p,t\right)dt$

$$\text{where } \lambda\left(p,t\right) = \int_t^\infty e^{-\delta(s-t)}\zeta\left(p,s\right)ds$$

Proof.

$$\begin{aligned}
& E\left\{\Pi_i|p,\phi_i\right\} - w_i\left(0\right) \\
= & \int_0^\infty e^{-\delta s}\zeta\left(p,s\right)a_i\left(t|\phi_i,p\right)ds \quad \text{by lemma 4} \\
= & \int_0^\infty e^{-\delta s}\zeta\left(p,s\right)\left[a_i\left(0\right) + \int_0^s \phi_i\left(p_t\right)dt\right]ds \\
= & a_i\left(0\right)\lambda\left(p,0\right) + \int_0^\infty e^{-\delta s}\zeta\left(p,s\right)\int_0^s \phi_i\left(p_t\right)dt\,ds \\
= & a_i\left(0\right)\lambda\left(p,0\right) + \int_0^\infty \phi_i\left(p_t\right)\int_t^\infty e^{-\delta s}\zeta\left(p,s\right)ds\,dt \\
= & a_i\left(0\right)\lambda\left(p,0\right) + \int_0^\infty e^{-\delta t}\phi_i\left(p_t\right)\lambda\left(p,t\right)dt
\end{aligned}$$

□

Lemma 6. $\tilde{p}\left(t\right) = p\left(t\right) + \int_t^\infty e^{-\delta(t-x)}\dot{p}\left(x\right)dx$

Proof.

$$\begin{aligned}
\tilde{p}\left(t\right) &= \delta\int_t^\infty e^{-\delta(s-t)}p\left(s\right)ds \\
&= p\left(t\right) + \delta\int_t^\infty e^{-\delta(s-t)}\int_t^s \dot{p}\left(x\right)dx\,ds \\
&= p\left(t\right) + \delta e^{\delta t}\int_t^\infty \dot{p}\left(x\right)\int_x^\infty e^{-\delta s}ds\,dx \\
&= p\left(t\right) + \delta e^{\delta t}\int_t^\infty \dot{p}\left(x\right)\left[\frac{e^{-\delta x}}{\delta}\right]dx \\
&= p\left(t\right) + \int_t^\infty e^{-\delta(x-t)}\dot{p}\left(x\right)dx
\end{aligned}$$

□

Lemma 7. *$\lambda(p,t) = V(p,t) - p(t)$ where $V(p,t) = \alpha\bar{p} + (1-\alpha)\tilde{p}(t)$*

Proof.

$$\begin{aligned}
\lambda(p,t) &= \int_t^\infty e^{-\delta(s-t)}\zeta(p,s)\,ds \\
&= \int_t^\infty e^{-\delta(s-t)}\left[z + \dot{p}(t) - rp(t)\right]ds \\
&= \frac{z}{\delta} + \int_t^\infty e^{-\delta(s-t)}\dot{p}(t)\,ds - r\int_t^\infty e^{-\delta(s-t)}p(t)\,ds \\
&= \frac{z}{\delta} + \tilde{p}_t - p(t) - \left(\frac{r}{\delta}\right)\tilde{p}(t) \qquad \text{by lemma 6} \\
&= \left(\frac{r}{\delta}\right)\frac{z}{r} + \left(1 - \frac{r}{\delta}\right)\tilde{p}(t) - p(t) \\
&= \alpha\bar{p} + (1-\alpha)\tilde{p}(t) - p(t) \\
&= V(p,t) - p(t)
\end{aligned}$$

□

Lemma 8. *$\phi_i^* \in \operatorname*{argmax}_{\phi_i\in\Phi_i} E\{\Pi_i|p,\phi_i\}$ where*

$$\phi_i^*(p_t) = \gamma\ \mathrm{sgn}\left(V(p,t) - p(t)\right)$$

Proof. By lemma 5

$$E\{\Pi_i|p,\phi_i\} = w_i(0) + a_i(0)\,\lambda(p,0) + \int_0^\infty e^{-\delta t}\phi_i(p_t)\,\lambda(p,t)\,dt$$

By lemma 7

$$E\{\Pi_i|p,\phi_i\} = w_i(0) + a_i(0)\,\lambda(p,0) + \int_0^\infty e^{-\delta t}\phi_i(p_t)\left[V(p,t) - p(t)\right]dt$$

Hence ϕ_i^* maximizes $E\{\Pi_i|p,\phi_i\}$. □

Lemma 9. *There exists $\phi^* \in \Phi$ and $p^* \in P$ such that*

$$\phi_i^*(p_t) = \gamma \operatorname{sgn}(\bar{p} - p(t))$$
$$t^* = \gamma^{-1} \left| f^{-1}(\bar{p}) - A(0) \right|$$
$$p^*(t) = f(A(t|\phi^*, p^*)) = \begin{cases} f(A(0) + \operatorname{sgn}(\bar{p} - p^*(0))\gamma t) & \text{if } t < t^* \\ \bar{p} & \text{if } t \geq t^* \end{cases}$$
$$\phi_i^* \in \underset{\phi_i \in \Phi_i}{\operatorname{argmax}}\ E\{\Pi_i | p^*, \phi_i\}$$

Proof. Differentiating the market clearing condition obtains

$$p^*(t) = f(A(t|\phi^*, p^*))$$
$$\dot{p}^*(t) = \phi_i^*(p_t) f'\left(f^{-1}(p(t))\right)$$
$$\dot{p}^*(t) = \gamma \operatorname{sgn}(\bar{p} - p(t)) f'\left(f^{-1}(p(t))\right)$$

If $t \geq t^*$ then $p^*(t) = \bar{p} = \frac{z}{r}$ and

$$\lambda(p^*, t) = \int_t^\infty e^{-\delta(s-t)} \zeta(p^*, s)\, ds$$
$$\lambda(p^*, t) = \int_t^\infty e^{-\delta(s-t)} [z + \dot{p}^*(t) - rp^*(t)]\, ds$$
$$\lambda(p^*, t) = \int_t^\infty e^{-\delta(s-t)} [z + 0 - z]\, ds = 0$$

If $t < t^*$ then

$$\lambda(p^*, t) = \int_t^{t^*} e^{-\delta(s-t)} [z + \dot{p}^*(t) - rp^*(t)]\, ds$$
$$\lambda(p^*, t) = \int_t^{t^*} e^{-\delta(s-t)} r \left[\frac{z}{r} + \frac{1}{r}\dot{p}^*(t) - p^*(t)\right] ds$$
$$\lambda(p^*, t) = \int_t^{t^*} e^{-\delta(s-t)} r \left[\bar{p} - p^*(t) + \frac{1}{r}\dot{p}^*(t)\right] ds$$
$$\operatorname{sgn}(\lambda(p^*, t)) = \operatorname{sgn}(\bar{p} - p^*(t)) = \operatorname{sgn}(\bar{p} - p^*(t)) = \operatorname{sgn}(\phi_i^*(p_t^*))$$

Hence $\phi_i^* \in \underset{\phi_i \in \Phi_i}{\operatorname{argmax}}\ E\{\Pi_i | p^*, \phi_i\}$ by lemma 7 and lemma 8. □

Proof of theorem 1. Maximizing agent i's expected payoff conditional on the price history $p_s \in \mathcal{H}$ obtains

$$\begin{aligned}
& \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}} \int_P E\left\{\Pi_i | \varphi_i, p\right\} d\mu_i\left(p | p_s\right) \\
= & \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}}\, E\left\{E\left\{\Pi_i | \varphi_i, p\right\} \mid p_s\right\} \\
= & \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}}\, E\left\{\int_0^\infty e^{-\delta t} \phi_i\left(p_t\right) \lambda\left(p, t\right) dt \,\middle|\, p_s\right\} \quad \text{by Lemma 5} \\
= & \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}} \int_0^\infty e^{-\delta t} E\left\{\phi_i\left(p_t\right) \lambda\left(p, t\right) | \, p_s\right\} dt \\
= & \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}} \int_0^\infty e^{-\delta t} E\left\{\phi_i\left(p_t\right) E\left\{\lambda\left(p, t\right) | \, p_t\right\} | \, p_s\right\} dt \\
= & \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}} \int_0^\infty e^{-\delta t} E\left\{\phi_i\left(p_t\right) E\left\{V\left(p, t\right) - p\left(t\right) | \, p_t\right\} | \, p_s\right\} dt \quad \text{by Lemma 7} \\
= & \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}} \int_0^\infty e^{-\delta t} E\left\{\phi_i\left(p_t\right) \left[E\left\{V\left(p, t\right) | \, p_t\right\} - p\left(t\right)\right] | \, p_s\right\} dt \\
= & \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}} \int_0^\infty e^{-\delta t} E\left\{\phi_i\left(p_t\right) \left[V\left(p_t, \mu_i\right) - p\left(t\right)\right] | \, p_s\right\} dt \\
= & \underset{\varphi_i \in \phi_{is}^+}{\operatorname{argmax}}\, E\left\{\int_0^\infty e^{-\delta t} \phi_i\left(p_t\right) \left[V\left(p_t, \mu_i\right) - p\left(t\right)\right] dt \,\middle|\, p_s\right\}
\end{aligned}$$

Hence ϕ_i maximizes agent i's expected payoff conditional on the price history $p_s \in \mathcal{H}$ if and only if almost surely for almost all $t > s$

$$\phi_i\left(p_t\right) = \begin{cases} \gamma & \text{if } p\left(t\right) < V\left(p_t, \mu_i\right) \\ -\gamma & \text{if } p\left(t\right) > V\left(p_t, \mu_i\right) \end{cases}$$

□

Lemma 10. *There exists* $\check{p}_0 < \bar{p} < \hat{p}_0$ *such that*

$$p(t) < \check{p}_0 \implies V(p_t, \mu_i) > p(t)$$
$$p(t) > \hat{p}_0 \implies V(p_t, \mu_i) < p(t)$$

Proof. By theorem 1

$$\begin{aligned} V(p_t, \mu_i) &= \alpha \bar{p} + (1-\alpha) \int_P \tilde{p}(t)\, d\mu_i(p|p_t) \\ &= \alpha \bar{p} + (1-\alpha) \int_P \delta \int_0^\infty e^{-\delta s} p(t+s)\, ds\, d\mu_i(p|p_t) \end{aligned}$$

Then by lemma 2

$$\begin{aligned} V(p_t, \mu_i) &= \alpha \bar{p} + (1-\alpha)\, \delta \int_0^\infty e^{-\delta s} \left[p(t) + O(s^c)\right] ds \\ V(p_t, \mu_i) &= \alpha \bar{p} + (1-\alpha) \left[p(t) + O(1)\right] \\ \frac{V(p_t, \mu_i)}{p(t)} &= \alpha \frac{\bar{p}}{p(t)} + (1-\alpha) \left[1 + \frac{O(1)}{p(t)}\right] \\ \lim_{p(t)\to\infty} \frac{V(p_t, \mu_i)}{p(t)} &= \lim_{p(t)\to-\infty} \frac{V(p_t, \mu_i)}{p(t)} = (1-\alpha) \in (0,1) \end{aligned}$$

Hence there exists $\check{p}_0 < \bar{p} < \hat{p}_0$ such that

$$p(t) < \check{p}_0 \implies V(p_t, \mu_i) > p(t)$$
$$p(t) > \hat{p}_0 \implies V(p_t, \mu_i) < p(t)$$

□

Lemma 11. *If $p \in P_\infty$ then $p(t) \leq p^*(t)$*

Proof. If $p \in P_0$ and $p(t) > \hat{p}_0$ then

$$\begin{aligned} V(p_t, \mu_i) &< p(t) \quad \text{by lemma 10} \\ \phi_i(p_t) &= -\gamma \quad \text{by theorem 1} \\ \dot{p}(t) &= -\gamma f'\left(f^{-1}(p(t))\right) < 0 \end{aligned}$$

For all $n \in \mathbb{N}$ let $\hat{p}_n = \alpha\bar{p} + (1-\alpha)\hat{p}_{n-1}$ so that

$$\hat{p}_n = \alpha\bar{p}\sum_{k=0}^{n-1}(1-\alpha)^k + (1-\alpha)^n \hat{p}_0 \quad \text{for all } n \in \mathbb{N}$$

For the inductive hypothesis, suppose

$$p(t) > \hat{p}_n \text{ and } p \in P_n \implies \dot{p}(t) = -\gamma f'\left(f^{-1}(p(t))\right)$$

Now if $p(t) < \hat{p}_n$ and $p \in P_n$ then

$$\begin{aligned} V(p,t) &= \alpha\bar{p} + (1-\alpha)\tilde{p}(t) \\ V(p,t) &\leq \alpha\bar{p} + (1-\alpha)\hat{p}_n = \hat{p}_{n+1} \\ V(p_t, \mu_i) &\leq \hat{p}_{n+1} \quad \text{for all } \mu \in M^{n+1} \\ p(t) > \hat{p}_{n+1} \text{ and } p \in P_{n+1} &\implies \dot{p}(t) = -\gamma f'\left(f^{-1}(p(t))\right) \end{aligned}$$

Hence by induction

$$p(t) > \hat{p}_n \implies \dot{p}(t) = -\gamma f'\left(f^{-1}(p(t))\right) \quad \text{for all } n \in \mathbb{N}$$

Taking the limit as $n \to \infty$ obtains

$$\lim_{n\to\infty}\hat{p}_n = \alpha\bar{p}\sum_{k=0}^{\infty}(1-\alpha)^k = \bar{p}$$

$$p(t) > \bar{p} \text{ and } p \in P_\infty \implies \dot{p}(t) = -\gamma f'\left(f^{-1}(p(t))\right)$$

Hence $p(t) \leq p^*(t)$ for all $p \in P_\infty$. □

Lemma 12. *If $p \in P_\infty$ then $p(t) \geq p^*(t)$*

Proof. If $p \in P_0$ and $p(t) < \check{p}_0$ then

$$\begin{aligned} V(p_t, \mu_i) &> p(t) \quad \text{by lemma 10} \\ \phi_i(p_t) &= \gamma \quad \text{by theorem 1} \\ \dot{p}(t) &= \gamma f'\left(f^{-1}(p(t))\right) < 0 \end{aligned}$$

For all $n \in \mathbb{N}$ let $\check{p}_n = \alpha\bar{p} + (1-\alpha)\check{p}_{n-1}$ so that

$$\check{p}_n = \alpha\bar{p}\sum_{k=0}^{n-1}(1-\alpha)^k + (1-\alpha)^n\check{p}_0 \quad \text{for all } n \in \mathbb{N}$$

For the inductive hypothesis, suppose

$$p(t) < \check{p}_n \text{ and } p \in P_n \implies \dot{p}(t) = \gamma f'\left(f^{-1}(p(t))\right)$$

Now if $p(t) > \check{p}_n$ and $p \in P_n$ then

$$\begin{aligned} V(p,t) &= \alpha\bar{p} + (1-\alpha)\tilde{p}(t) \\ V(p,t) &\geq \alpha\bar{p} + (1-\alpha)\check{p}_n = \check{p}_{n+1} \\ V(p_t, \mu_i) &\geq \check{p}_{n+1} \quad \text{for all } \mu \in M^{n+1} \\ p(t) < \check{p}_{n+1} \text{ and } p \in P_{n+1} &\implies \dot{p}(t) = \gamma f'\left(f^{-1}(p(t))\right) \end{aligned}$$

Hence by induction

$$p(t) < \check{p}_n \text{ and } p \in P_n \implies \dot{p}(t) = \gamma f'\left(f^{-1}(p(t))\right) \quad \text{for all } n \in \mathbb{N}$$

Taking the limit as $n \to \infty$ obtains

$$\lim_{n\to\infty}\check{p}_n = \alpha\bar{p}\sum_{k=0}^{\infty}(1-\alpha)^k = \alpha\bar{p}\left(\frac{1}{\alpha}\right) = \bar{p}$$

$$p(t) < \bar{p} \text{ and } p \in P_\infty \implies \dot{p}(t) = \gamma f'\left(f^{-1}(p(t))\right)$$

Hence $p(t) \leq p^*(t)$ for all $p \in P_\infty$. □

Lemma 13. $P_\infty = \{p^*\}$

Proof. Let $g : P \times \mathbb{R}_+ \to P$ such that

$$g(p,t)(s) = \begin{cases} p(s) & \text{if } s \le t \\ f\left(f^{-1}(p(t)) + \gamma \operatorname{sgn}(\bar{p} - p(t))(s-t)\right) & \text{if } t \le s \le \tau \\ \bar{p} & \text{if } s \ge \tau \end{cases}$$

$$\tau = \gamma^{-1}\left|f^{-1}(p(t)) - \bar{p}\right|$$

By lemma 9, $\phi_i^* \in \underset{\phi_i \in \Phi_i}{\operatorname{argmax}}\ E\{\Pi_i | p^*, \phi_i\}$. Let $\mu_i \in M$ such that

$$\mu_i\left(g(p,t) \,|\, p_t\right) = 1$$

Hence by theorem 1

$$V(p_t, \mu_i) = \alpha\bar{p} + (1-\alpha)\int \tilde{p}(t)\, d\mu_i(p|p_t)$$

$$V(p_t, \mu_i) = \alpha\bar{p} + (1-\alpha)\delta\int_0^\infty e^{-\delta s}\int p(t+s)\, d\mu_i(p|p_t)\, ds$$

$$V(p_t, \mu_i) = \alpha\bar{p} + (1-\alpha)\delta\int_0^\infty e^{-\delta s} g(p,t)(t+s)\, ds$$

$$\phi_i^*(p_t) = \gamma \operatorname{sgn}(\bar{p} - p(t)) = \gamma \operatorname{sgn}(V(p_t, \mu_i) - p(t))$$

$$\phi_i^* \in \underset{\varphi_i \in \phi_{it}^+}{\operatorname{argmax}} \int E\{\Pi_i | \varphi_i, p\}\, d\mu_i(p|p_t) \quad \text{for all } p_t \in \mathcal{H}$$

By lemma 9, $p^*(t) = f(A(t|\phi^*, p^*))$ so $p^* \in P_0$. Continuing this way, $p^* \in P_n$ for all $n \in \mathbb{N}$. By lemma 11, $p(t) \le p^*(t)$ for all $p \in P_\infty$. By lemma 12, $p(t) \ge p^*(t)$ for all $p \in P_\infty$. Hence $P_\infty = \{p^*\}$. □

Lemma 14. *For all $n \in \mathbb{N}$ there exists $p \in P_n$ such that $p \neq p^*$*

Proof. Let $t^* = \gamma^{-1} \left| f^{-1}(\bar{p}) - f^{-1}(p(0)) \right|$. Let $k = \gamma$ if $p(0) \leq \bar{p}$ and $k = -\gamma$ otherwise. Let $h : \mathbb{R}_+ \to P$ such that

$$h(c)(s) = \begin{cases} f\left(f^{-1}(p(0)) + kt\right) & \text{if } t < t^* + c \\ \bar{p} & \text{if } t \geq t^* + c \end{cases}$$

Let $m : P \to M$ such that

$$\begin{aligned} m_i(y)(y|p_t) &= 1 \quad \text{if } p_t = y_t \\ m_i(y)(g(p,t) \mid p_t) &= 1 \quad \text{if } p_t \neq y_t \end{aligned}$$

If $y = h(1)$ and $\mu = m(y)$ then by theorem 1

$$\begin{aligned} V(y_t, \mu_i) &= \alpha \bar{p} + (1-\alpha) \int \tilde{p}(t)\, d\mu_i(p|y_t) \\ &= \alpha \bar{p} + (1-\alpha)\, \delta \int_0^\infty e^{-\delta s} y(t+s)\, ds \end{aligned}$$

If $y(0) \leq \bar{p}$ then y is increasing over $[0, t^*+1]$, decreasing over $[t^*+1, t^*+2]$, and constant at $\bar{p}$ for $t \geq t^*+2$, so there exists $c_0 \in (0,1)$ such that $V(y_t, \mu_i) \geq y(t)$ for $t < t^* + c_0$ and $V(y_t, \mu_i) \leq y(t)$ for $t > t^* + c_0$. Conversely if $y(0) > \bar{p}$ then y is decreasing over $[0, t^*+1]$, increasing over $[t^*+1, t^*+2]$, and constant for $t \geq t^*+2$, so there exists $c_0 \in (0,1)$ such that $V(y_t, \mu_i) \leq y(t)$ for $t < t^*+c_0$ and $V(y_t, \mu_i) \leq y(t)$ for $t > t^*+c_0$. By theorem 1, if $\phi_i(p_t) = \gamma \operatorname{sgn}(V(p_t, \mu_i))$ then ϕ_i is rational under μ_i. If $y_0 = h(c_0)$ then $y_0(t) = f(A(t|\phi_i, y_0))$, so $y_0 \in P_0$. By the same argument, there exists $c_1 \in (0, c_0)$ such that $h(c_1) \in P_1$. Continuing this way, there exists $c_n > 0$ such that $h(c_n) \in P_n$ for all $n \in \mathbb{N}$. ☐

Proof of theorem 2. By lemma 9, p^* is rationalizable. By lemma 13, $P_\infty = \{p^*\}$. By lemma 14, there exists $p \in P_n$ such that $p \neq p^*$ for all $n \in \mathbb{N}$. ☐