Dynamic Investment in Durable Equipment

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Abstract

This paper studies a continuous time game in which a finite number of investors purchase durable equipment on a primary market and rent it out, exercising market power as oligopolists in the rental market and as oligopsonists in the purchase market. New equipment is supplied along an upward sloping curve, so the purchase price rises with the aggregate purchase rate. The unique open loop equilibrium is characterized in closed form. Even though the model has no internal adjustment cost, capacity constraint, or price rigidity, the aggregate equipment stock adjusts gradually toward the steady state. Gradual adjustment arises from the endogenous purchase price alone, and the adjustment speed is an equilibrium object rather than an imposed parameter. The open loop equilibrium path is supported by a Markov perfect equilibrium, under which investors are not required to fix their investment path in advance.

1 Introduction

Durable equipment such as computer servers, aircraft, shipping containers, and construction machinery is often concentrated in the hands of a small number of investors who purchase it on a primary market and rent it out. Such investors may exert market power in both the rental market and the purchase market. As oligopolists in the rental market, they may restrain the supply of equipment to increase the rents they earn. As oligopsonists in the purchase market, they may restrain the rate at which they purchase equipment to lower the prices they pay.

This issue is empirically grounded. Gavazza (2011) documents the prominence of leasing in the market for commercial aircraft. Gurun et al. (2023) find that large institutional landlords exercise market power in rental housing. Because equipment is durable, today's purchases determine tomorrow's rental supply, so the rental margin and the purchase margin are linked over time. This paper studies that linkage in a continuous time game of equipment accumulation.

In the model, a finite number of investors begin with arbitrary initial endowments and choose purchase rates over time. The rental rate is decreasing in the aggregate equipment stock, and the purchase price is increasing in the aggregate purchase rate. Equipment depreciates continuously over time. The only adjustment friction is the upward sloping supply curve for new equipment. Purchasing equipment more rapidly raises the purchase price, so the aggregate equipment quantity adjusts gradually. There are no internal adjustment costs, capacity constraints, or exogenous bounds on strategies.

The first result characterizes the unique open loop equilibrium in closed form. Even if initial endowments are asymmetric, every investor holds the same amount of equipment in the steady state. The total equipment stock in the steady state is equal to the competitive equipment stock marked down by a Cournot factor which reflects market power in both the rental market and the purchase market. In the unique open loop equilibrium, investors adjust their equipment holdings at a common rate. The second result shows that the open loop equilibrium path is supported by a Markov perfect equilibrium in which investors are not required to fix their investment path in advance.

Existing models of dynamic oligopoly typically generate gradual adjustment through some combination of three devices: internal adjustment costs, exogenous constraints on strategies, or exogenous price rigidities. In contrast, the present paper generates gradual adjustment through the price mechanism itself. The equipment price is determined by an upward sloping supply curve, so it depends on the aggregate purchase rate.

In the classical capital accumulation games of Reynolds (1987), Driskill and McCafferty (1989), and Reynolds (1991), firms face a convex internal adjustment cost. The convexity of this adjustment cost function is the source of their gradual dynamics. In contrast, the present model has no internal adjustment cost.

Grenadier (2002), Back and Paulsen (2009), and Steg (2024) consider models where capital is acquired at a constant exogenous marginal cost. In these models, capital never depreciates and investment is irreversible, so the capital stock is always nondecreasing. In the models of Spence (1979) and Fudenberg and Tirole (1983), investment rates are exogenously bounded and investments are irreversible. In the present paper, investment is reversible and unbounded, equipment depreciates, and the price of equipment is endogenous.

In these prior models, open loop behavior generally lacks credibility if investors are not required to fix their investment path in advance. In the models presented by Reynolds (1987), Driskill and McCafferty (1989), and Reynolds (1991), Markov perfect equilibria are more competitive than open loop equilibria because firms overaccumulate as a way of incentivizing others to accumulate less. Back and Paulsen (2009) show that open loop equilibria are not subgame perfect in their model. Steg (2024) identifies a range of Markov perfect equilibria, none of which support the open loop equilibrium path.

Dockner, Plank, and Nishimura (1999) provide a precedent for the coincidence of these two solution concepts in a discrete time model with a fixed exogenous capital price. In their model, capital jumps to its steady state immediately, so the coincidence reflects the absence of adjustment dynamics. In the present model, the equipment price is endogenous, time is continuous, and the open loop equilibrium is supported by Markov strategies along a gradual transition path.

The models presented by Fershtman and Kamien (1987) and Wang and Werning (2022) generate dynamics from price rigidities. Fershtman and Kamien (1987) consider a model where output prices adjust towards inverse demand at an exogenous speed. Wang and Werning (2022) present a model where firms can only change prices at exogenously staggered times. In the present model, prices adjust freely at every instant, so the speed of adjustment is an equilibrium object with no exogenous constraint.

2 Model

Consider a population of n investors who can purchase and rent out durable equipment. Let $N = \{1, \ldots, n\}$ index the set of investors. At time $t = 0$, each investor i is endowed with $q_{0i} \in \mathbb{R}$ units of equipment. Let $q_0 \in \mathbb{R}^n$ denote the initial endowment profile.

Let $q_i(t)$ denote the quantity of equipment owned by investor i at time $t \in \mathbb{R}_+$. Let $q(t) \in \mathbb{R}^n$ denote the quantity profile at time t. Let $x_i(t) \in \mathbb{R}$ denote investor i's net purchase rate at time t. Let $x(t) \in \mathbb{R}^n$ denote the purchase profile at time t. Let $\delta > 0$ denote the depreciation rate, so $\dot{q}_i(t) = x_i(t) - \delta q_i(t)$.

Investors are oligopolists in the rental market and oligopsonists in the purchase market. Let $Q(t) = \sum_{i=1}^{n} q_i(t)$ denote the total quantity of equipment supplied to the rental market at time t. Let $z(t) = \gamma - \eta Q(t)$ denote the rental rate at time t. Supplying an additional unit of equipment to the rental market decreases the rental rate, so the marginal rental revenue is less than the rental rate.

Let $X(t) = \sum_{i=1}^{n} x_i(t)$ denote the aggregate purchase rate at time t. Let $p(t) = \alpha + \beta X(t)$ denote the purchase price at time t. New equipment is supplied competitively with an upward sloping supply curve. The faster investors buy equipment, the higher the purchase price, so the slope β acts as an aggregate adjustment friction.

Let $\pi_i(t) = z(t)\, q_i(t) - p(t)\, x_i(t)$ denote investor i's net cash flow at time t. Let $r > 0$ denote the discount rate. The parameters α, β, δ and η are strictly positive and $\gamma > \alpha(r + \delta)$. This restriction guarantees that the equipment market is viable. A unit of equipment purchased at the base price α has an opportunity cost of $\alpha(r + \delta)$ per unit time in forgone interest and depreciation. The restriction $\gamma > \alpha(r + \delta)$ says that the base rent γ exceeds this opportunity cost.

Let $\mathcal{X}$ denote the set of locally square-integrable functions $f : \mathbb{R}_+ \to \mathbb{R}$. Let $x \in \mathcal{X}^n$ denote the investment profile. Let $\Pi_i(x|q_0) = \int_0^\infty e^{-rt} \pi_i(t)\, dt \in \overline{\mathbb{R}}$ denote investor i's total discounted cash flow when this limit exists in the extended reals. Otherwise, let $\Pi_i(x|q_0) = -\infty$.[1]

[1] This convention assigns an infinitely negative value to oscillatory non-convergent cases. Without it, such cases would have no well-defined value.

3 Results

An investment profile $x \in \mathcal{X}^n$ and an initial endowment profile $q_0 \in \mathbb{R}^n$ are said to form an open loop equilibrium (x, q_0) if $\Pi_i(x_i, x_{-i}|q_0) \geq \Pi_i(\hat{x}_i, x_{-i}|q_0)$ for all $i \in N$ and all $\hat{x}_i \in \mathcal{X}$. In an open loop equilibrium, each investor $i \in N$ commits to a purchase path x_i at time $t = 0$ taking rival paths x_{-i} as given. An open loop equilibrium is thus a Nash equilibrium in purchase path strategies. Theorem 1 identifies a unique open loop equilibrium for each initial endowment profile.

Theorem 1. *For every initial endowment profile $q_0 \in \mathbb{R}^n$ there exists a unique $x \in \mathcal{X}^n$ such that (x, q_0) is an open loop equilibrium. For every investor $i \in N$ and all $t \in \mathbb{R}_+$*

$$x_i(t) = \delta\bar{q} + (\delta + \kappa)(q_{0i} - \bar{q}) e^{\kappa t} \tag{1}$$

$$\bar{q} = \left(\frac{1}{n+1}\right)\left[\frac{\gamma - \alpha(r+\delta)}{\eta + \beta\delta(r+\delta)}\right] \tag{2}$$

$$\kappa = \frac{r\beta - \sqrt{\beta^2(r+2\delta)^2 + 4\beta\eta}}{2\beta} \tag{3}$$

Even if the initial endowments $q_0 \in \mathbb{R}^n$ are asymmetric, the steady state is symmetric. All investors hold $\bar{q}$ units of equipment in the steady state. Investors interact through the purchase price p and the rental rate z, but each investor's gap $q_i - \bar{q}$ decays at the common rate κ, regardless of how the initial endowments were distributed. The bracketed term in (2) is the aggregate equipment quantity that price taking investors would hold in the steady state. The factor $1/(n+1)$ is the familiar Cournot ratio of individual output to the competitive aggregate, here reflecting market power in both the rental and purchase markets.

In the open loop equilibrium, investor i's equipment quantity follows $q_i = \bar{q} + (q_{0i} - \bar{q}) e^{\kappa t}$, so their purchase rate follows $x_i = \delta q_i + \kappa(q_i - \bar{q})$. The first term δq_i offsets depreciation and the second term $\kappa(q_i - \bar{q})$ gradually closes the gap between q_i and $\bar{q}$ at a rate proportional to the size of the gap. The adjustment rate κ is faster when β is small. In the limit as $\beta \to 0$, the purchase price becomes insensitive to the aggregate purchase rate and the adjustment rate κ grows without bound.

Let Φ denote the set of twice continuously differentiable globally Lipschitz functions $f : \mathbb{R}^n \to \mathbb{R}$. Let $\phi_i \in \Phi$ denote investor i's strategy. Let $\phi \in \Phi^n$ denote the strategy profile. Let $\Pi_i(\phi|q_0) = \Pi_i(x|q_0)$ where $x(t) = \phi(q(t))$. Let $\Pi_i(\hat{x}_i, \phi_{-i}|q_0) = \Pi_i(x|q_0)$

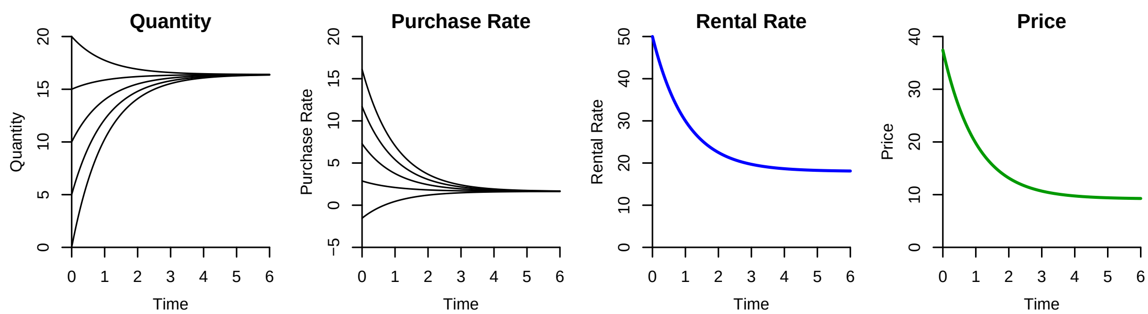

Figure 1: An Equilibrium Path

where $x_i = \hat{x}_i$ and $x_{-it} = \phi_{-i}(q_t)$.[2] A strategy profile $\phi \in \Phi^n$ is said to be a Markov perfect equilibrium if $\Pi_i(\phi_i, \phi_{-i}|q_0) \geq \Pi_i(\hat{x}_i, \phi_{-i}|q_0)$ for all $i \in N$, $\hat{x}_i \in \mathcal{X}$, and $q_0 \in \mathbb{R}^n$. Theorem 2 identifies a Markov perfect equilibrium under which each investor's purchase rate depends only on their own equipment quantity.

Theorem 2. *There exists a Markov perfect equilibrium $\phi \in \Phi^n$ such that $\phi_i(q) = \delta q_i + \kappa(q_i - \bar{q})$ for every investor $i \in N$ and every quantity profile $q \in \mathbb{R}^n$.*

Under this Markov perfect equilibrium, the investment path from any initial endowment profile $q_0 \in \mathbb{R}^n$ coincides with the unique open loop equilibrium. Under investor i's Markov strategy, their purchase rate depends on their own equipment quantity, but not on the quantity held by others. Likewise, rival strategies do not respond to investor i's equipment quantity, so a deviation from investor i's Markov strategy cannot influence the future behavior of rivals. This independence is what reconciles the open loop equilibrium path with a Markov perfect equilibrium.

4 Example

Figure 1 illustrates the equilibrium path when $n = 5$, $r = 0.05$, $\delta = 0.1$, $\alpha = \beta = \eta = 1$, and $\gamma = 100$. The initial endowments at time $t = 0$ are $q_0 = (0, 5, 10, 15, 20)$. Over time, each investor's individual equipment quantity converges to the common steady state level $\bar{q} \approx 16.4$. Initial endowments are asymmetric, but steady state equipment holdings are symmetric. The equipment price and the rental rate decline over time as equipment quantities approach the steady state. Investor 5 starts with equipment quantity $q_{05} = 20 > \bar{q}$, so their equipment quantity declines over time and their initial

[2]The fully closed loop initial value problem has a unique global solution because ϕ_i is globally Lipschitz for each investor i. Further, the mixed closed loop initial value problem has a unique global solution because ϕ_j is globally Lipschitz for $j \neq i$ and $\hat{x}_i$ enters as a Carathéodory forcing.

purchase rate is negative, illustrating the reversibility of investment in this model.

5 Conclusion

This paper studies a game in which a finite number of investors accumulate durable equipment in continuous time. These investors exercise market power as oligopolists in the rental market and as oligopsonists in the purchase market. In this model, gradual adjustment arises from the price mechanism itself. New equipment is supplied along an upward sloping curve, so the purchase price depends on the aggregate purchase rate. This dependence alone is enough to make the aggregate equipment stock adjust gradually. No internal adjustment cost, capacity constraint, or exogenous price rigidity is required. The speed of adjustment is an equilibrium object rather than an imposed parameter.

The open loop equilibrium is characterized in closed form and is supported by a Markov perfect equilibrium along the same path. This coincidence rests on the linear quadratic structure of the model. In equilibrium, each investor's equipment quantity evolves independently of the quantities held by rivals, so the strategic feedback that ordinarily separates Markov perfect from open loop behavior is absent. Whether this coincidence can survive when demand or supply relations are nonlinear remains an open question.

Theorem 2 establishes the existence of a Markov perfect equilibrium supporting the open loop path, not the uniqueness of this equilibrium. The Markov strategies considered here condition only on an investor's own equipment quantity. Equilibria in which investors condition on the full quantity profile are not ruled out. In related capital accumulation games, such feedback equilibria are more competitive than the open loop equilibrium. Characterizing the full set of Markov perfect equilibria in this model is left for future work.

Several extensions of the present model seem worth pursuing. The linear rental demand and equipment supply could be replaced by nonlinear schedules. Allowing the demand and cost parameters to differ across investors could allow heterogeneity to survive in the steady state. If the deterministic rental demand relation were enriched to include a stochastic component, the endogenous purchase price could become a channel for dynamic strategic interaction beyond the one studied here. Each of these extensions would relax a structural assumption on which the present results depend.

A Proofs

Lemma 1. *Investor i's net cash flow π_i is continuously differentiable and jointly strictly concave in (x_i, q_i).*

Proof. Differentiating π_i with respect to x_i and q_i obtains (4) and (5).

$$\frac{\partial \pi_i}{\partial x_i} = -p - \frac{\partial p}{\partial x_i} x_i = -\alpha - \beta (X + x_i) = -p - \beta x_i \tag{4}$$

$$\frac{\partial \pi_i}{\partial q_i} = z + \frac{\partial z}{\partial q_i} q_i = \gamma - \eta (Q + q_i) = z - \eta q_i \tag{5}$$

By (6)–(8), π_i is jointly strictly concave in (x_i, q_i).

$$\frac{\partial^2 \pi_i}{\partial x_i^2} = -2\beta < 0 \tag{6}$$

$$\frac{\partial^2 \pi_i}{\partial q_i^2} = -2\eta < 0 \tag{7}$$

$$\frac{\partial^2 \pi_i}{\partial x_i \partial q_i} = 0 \tag{8}$$

□

Lemma 2. *If (x, q_0) is an open loop equilibrium then $\Pi_i (x|q_0) > -\infty$.*

Proof. Let $\hat{q}_i : \mathbb{R}_+ \to \mathbb{R}$ such that for all $t \in \mathbb{R}_+$

$$\hat{q}_i (t) = \begin{cases} q_{0i} - t \operatorname{sgn}(q_{0i}) & t < |q_{0i}| \\ 0 & t \geq |q_{0i}| \end{cases}$$

Let $\hat{x}_i \in \mathcal{X}$ such that for all $t \in \mathbb{R}_+$

$$\hat{x}_i (t) = \begin{cases} \delta \hat{q}_i (t) - \operatorname{sgn}(q_{0i}) & \text{if } t < |q_{0i}| \\ 0 & \text{if } t \geq |q_{0i}| \end{cases}$$

If investor i were to follow $\hat{x}_i$, then q_i and x_i would both vanish after $t = |q_{0i}|$, so $\pi_i = z q_i - p x_i$ would also vanish after $t = |q_{0i}|$ and $\Pi_i (\hat{x}_i, x_{-i}|q_0)$ is finite. Since (x, q_0) is an open loop equilibrium, $\Pi_i (x|q_0) \geq \Pi_i (\hat{x}_i, x_{-i}|q_0) > -\infty$. □

Lemma 3. *If (x, q_0) is an open loop equilibrium then*

$$\sum_{i=1}^{n} \Pi_i (x|q_0) \leq \frac{1}{r} \left(\frac{\gamma^2}{4\eta} + \frac{\alpha^2}{4\beta} \right) \tag{9}$$

Proof. By the definition of π_i, z, and p

$$\begin{aligned}\sum_{i=1}^{n} \pi_i &= zQ - pX \\ &= (\gamma - \eta Q) Q - (\alpha + \beta X) X \\ &= \gamma Q - \eta Q^2 - \alpha X - \beta X^2 \\ &\leq \frac{\gamma^2}{4\eta} + \frac{\alpha^2}{4\beta} \end{aligned} \tag{10}$$

By Lemma 2, $\Pi_i (x|q_0) > -\infty$, so

$$\Pi_i (x|q_0) = \int_0^\infty e^{-rt} \pi_i dt \in \mathbb{R} \cup \{\infty\} \tag{11}$$

Summing over i produces

$$\begin{aligned}\sum_{i=1}^{n} \Pi_i (x|q_0) &= \int_0^\infty e^{-rt} \sum_{i=1}^{n} \pi_i dt \\ &\leq \int_0^\infty e^{-rt} \left(\frac{\gamma^2}{4\eta} + \frac{\alpha^2}{4\beta} \right) dt = \frac{1}{r} \left(\frac{\gamma^2}{4\eta} + \frac{\alpha^2}{4\beta} \right) \end{aligned} \tag{12}$$

□

Lemma 4. *If (x, q_0) is an open loop equilibrium then $\Pi_i (x|q_0)$ is finite.*

Proof. By Lemma 2, $\Pi_i (x|q_0) > -\infty$. By Lemma 3

$$\Pi_i (x|q_0) \leq \frac{1}{r} \left(\frac{\gamma^2}{4\eta} + \frac{\alpha^2}{4\beta} \right) - \sum_{j \neq i} \Pi_j (x|q_0) < \infty \tag{13}$$

□

Lemma 5. *If (x, q_0) is an open loop equilibrium then there exists $\lambda_i = p + \beta x_i$ such that $\dot{\lambda}_i = (r + \delta) \lambda_i - z + \eta q_i$.*

Proof. Fix $i \in N$ and treat x_{-i} as given, so investor i faces an infinite horizon control problem. Since (x, q_0) is an open loop equilibrium, $\Pi_i (x|q_0)$ is finite by Lemma 4. Since x_i is optimal over the infinite horizon and $\Pi_i (x|q_0)$ is finite, x_i is a finitely optimal control. The dynamics and π_i are continuously differentiable in (q_i, x_i), so assumption (A1) of Aseev and Veliov (2015) holds. Hence Theorem 2.1 of Aseev and Veliov (2015) applies. It yields a nonvanishing pair (ψ^0, ψ_i) with $\psi^0 \geq 0$ and present value Hamiltonian $H_i = \psi^0 e^{-rt} \pi_i + \psi_i (x_i - \delta q_i)$ solving the adjoint equation $\dot{\psi}_i = -\partial H_i / \partial q_i$ and maximized over x_i for almost all t. If ψ^0 were equal to zero, then H_i would be linear in x_i, but then maximizing H_i in x_i would require $\psi_i = 0$ and this would contradict the nonvanishing of (ψ^0, ψ_i), so ψ^0 must be strictly greater than zero and we normalize $\psi^0 = 1$. Now let $\lambda_i = e^{rt} \psi_i$, so the Hamiltonian becomes

$H_i = \pi_i + \lambda_i (x_i - \delta q_i)$ and the adjoint equation becomes $\dot{\lambda}_i = (r+\delta)\lambda_i - \partial \pi_i / \partial q_i$. The control x_i is unconstrained and the Hamiltonian is strictly concave in x_i by Lemma 1, so the maximum condition reduces to the interior first order condition $\partial H_i / \partial x_i = 0$. Using $\partial \pi_i / \partial x_i = -p - \beta x_i$ gives $\lambda_i = p + \beta x_i$. Using $\partial \pi_i / \partial q_i = z - \eta q_i$ in the adjoint equation gives $\dot{\lambda}_i = (r+\delta)\lambda_i - z + \eta q_i$. □

Lemma 6. *If* (x, q_0) *is an open loop equilibrium then there exists* $c \in \mathbb{R}^{2\times n}$ *such that*

$$q_i = \bar{q} + c_{1i} e^{\kappa t} + c_{2i} e^{\kappa_2 t} \tag{14}$$

$$\kappa = \frac{r\beta - \sqrt{\beta^2 (r+2\delta)^2 + 4\beta\eta}}{2\beta} < 0 \tag{15}$$

$$\kappa_2 = \frac{r\beta + \sqrt{\beta^2 (r+2\delta)^2 + 4\beta\eta}}{2\beta} > r + \delta \tag{16}$$

Proof. By Lemma 5, there exists λ_i such that

$$\lambda_i = p + \beta x_i \tag{17}$$

$$\dot{\lambda}_i = (r+\delta)\lambda_i - z + \eta q_i \tag{18}$$

Since λ_i is differentiable, so are p and x_i. Combining these two equations yields

$$\dot{p} + \beta \dot{x}_i = (r+\delta)(p + \beta x_i) - z + \eta q_i \tag{19}$$

Since $p = \alpha + \beta X$ and $z = \gamma - \eta Q$

$$\beta \dot{X} + \beta \dot{x}_i = (r+\delta)(\alpha + \beta X + \beta x_i) - \gamma + \eta Q + \eta q_i \tag{20}$$

Summing over $i \in N$ produces

$$\beta (n+1) \dot{X} = (r+\delta)(n\alpha + \beta (n+1) X) - n\gamma + \eta (n+1) Q \tag{21}$$

Dividing by $(n+1)$ obtains

$$\beta \dot{X} = (r+\delta)\left(\frac{n\alpha}{n+1} + \beta X\right) - \frac{n\gamma}{n+1} + \eta Q \tag{22}$$

Subtracting (22) from (20) yields

$$\beta \dot{x}_i = (r+\delta)\left(\frac{\alpha}{n+1} + \beta x_i\right) - \frac{\gamma}{n+1} + \eta q_i \tag{23}$$

Rearranging this produces

$$\beta \dot{x}_i - (r+\delta)\beta x_i - \eta q_i = \frac{(r+\delta)\alpha - \gamma}{n+1} \tag{24}$$

Since $x_i = \dot{q}_i + \delta q_i$

$$\beta \ddot{q}_i - r\beta \dot{q}_i - [\eta + \beta\delta (r+\delta)] q_i = \frac{(r+\delta)\alpha - \gamma}{n+1} \tag{25}$$

The characteristic equation is (26) with coefficients (27)–(29).

$$0 = A\kappa^2 + B\kappa + C \tag{26}$$

$$A = \beta \tag{27}$$

$$B = -r\beta \tag{28}$$

$$C = -\eta - \beta\delta\left(r + \delta\right) \tag{29}$$

The discriminant is strictly positive by (30).

$$B^2 - 4AC = \beta^2\left(r + 2\delta\right)^2 + 4\beta\eta > 0 \tag{30}$$

So the characteristic polynomial has two distinct real roots

$$\kappa = \frac{r\beta - \sqrt{\beta^2\left(r + 2\delta\right)^2 + 4\beta\eta}}{2\beta} < 0 \tag{31}$$

$$\kappa_2 = \frac{r\beta + \sqrt{\beta^2\left(r + 2\delta\right)^2 + 4\beta\eta}}{2\beta} > r + \delta \tag{32}$$

A particular solution to (25) is given by (33).

$$q_i = \left(\frac{1}{n+1}\right)\left[\frac{\gamma - \alpha\left(r + \delta\right)}{\eta + \beta\delta\left(r + \delta\right)}\right] = \bar{q} \tag{33}$$

Hence the general solution to (25) takes the form

$$q_i = \bar{q} + c_{1i}e^{\kappa t} + c_{2i}e^{\kappa_2 t}$$

$\square$

Lemma 7. *If (x, q_0) is an open loop equilibrium then $x_i = \delta\bar{q} + (\delta + \kappa)\left(q_{0i} - \bar{q}\right)e^{\kappa t}$.*

Proof. If (x, q_0) is an open loop equilibrium then by Lemma 6

$$q_i = \bar{q} + c_{1i}e^{\kappa t} + c_{2i}e^{\kappa_2 t} \tag{34}$$

Let $a = \delta + \kappa$ and $b = \delta + \kappa_2$ so

$$x_i = \delta q_i + \dot{q}_i = \delta\bar{q} + ac_{1i}e^{\kappa t} + bc_{2i}e^{\kappa_2 t} \tag{35}$$

Let $C_1 = \sum_{i=1}^n c_{1i}$ and $C_2 = \sum_{i=1}^n c_{2i}$, so summing over $i \in N$ produces

$$Q = n\bar{q} + C_1e^{\kappa t} + C_2e^{\kappa_2 t} \tag{36}$$

$$X = n\delta\bar{q} + aC_1e^{\kappa t} + bC_2e^{\kappa_2 t} \tag{37}$$

Summing $\pi_i = zq_i - px_i$ over $i \in N$ yields

$$\begin{aligned}\sum_{i=1}^n \pi_i &= zQ - pX = \gamma Q - \eta Q^2 - \alpha X - \beta X^2 \\ &= o\left(e^{2\kappa_2 t}\right) - \left(\eta + \beta b^2\right)C_2^2 e^{2\kappa_2 t}\end{aligned} \tag{38}$$

If $C_2 \neq 0$ then since $\kappa_2 > r + \delta$

$$\sum_{i=1}^{n} \Pi_i (x|q_0) = \int_0^\infty e^{-rt} \sum_{i=1}^{n} \pi_i dt = -\infty \tag{39}$$

But this contradicts Lemma 4, so we must have $C_2 = 0$ and

$$Q = n\bar{q} + C_1 e^{\kappa t} \tag{40}$$

$$X = n\delta\bar{q} + aC_1 e^{\kappa t} \tag{41}$$

Let $\hat{x}_i = x_i - \delta c_{2i}$. If $\hat{x} = (\hat{x}_i, x_{-i})$ then

$$\hat{q}_i = q_i - c_{2i}\left(1 - e^{-\delta t}\right) \tag{42}$$

$$\hat{z} = z + \eta c_{2i}\left(1 - e^{-\delta t}\right) \tag{43}$$

$$\hat{p} = p - \beta\delta c_{2i} \tag{44}$$

$$\hat{\pi}_i = \hat{z}\hat{q}_i - \hat{p}\hat{x}_i \tag{45}$$

$$\hat{\pi}_i - \pi_i = o\left(e^{\kappa_2 t}\right) + (\eta + \beta\delta b)\, c_{2i}^2 e^{\kappa_2 t} \tag{46}$$

Now if $c_{2i} \neq 0$ then since $\kappa_2 > r + \delta$

$$\Pi_i (\hat{x}_i, x_{-i}|q_0) - \Pi_i (x_i, x_{-i}|q_0) = \int_0^\infty e^{-rt} (\hat{\pi}_i - \pi_i)\, dt = +\infty \tag{47}$$

By Lemma 4, $\Pi_i (x|q_0)$ is finite, so $\Pi_i (\hat{x}_i, x_{-i}|q_0) = \infty > \Pi_i (x|q_0)$, but this contradicts the optimality of x_i in the open loop equilibrium (x, q_0). Hence $c_{2i} = 0$ so

$$q_i = \bar{q} + c_{1i} e^{\kappa t} \tag{48}$$

Substituting in $c_{1i} = q_{0i} - \bar{q}$ yields

$$q_i = \bar{q} + (q_{0i} - \bar{q})\, e^{\kappa t} \tag{49}$$

Differentiating with respect to t produces

$$\dot{q}_i = \kappa (q_{0i} - \bar{q})\, e^{\kappa t} = \kappa (q_i - \bar{q}) \tag{50}$$

Substituting $\dot{q}_i = x_i - \delta q_i$ into (50) obtains

$$x_i = \delta q_i + \kappa (q_i - \bar{q}) \tag{51}$$

Substituting (49) into (51) yields

$$x_i = \delta\bar{q} + (\delta + \kappa)(q_{0i} - \bar{q})\, e^{\kappa t} \tag{52}$$

$\square$

Definition 1. For $i \in N$, let

$$x_i^* = \delta \bar{q} + (\delta + \kappa)(q_{0i} - \bar{q}) e^{\kappa t} \tag{53}$$

$$q_i^* = \bar{q} + (q_{0i} - \bar{q}) e^{\kappa t} \tag{54}$$

$$X^* = \sum_{i=1}^{n} x_i^* \qquad Q^* = \sum_{i=1}^{n} q_i^* \tag{55}$$

$$p^* = \alpha + \beta X^* \qquad z^* = \gamma - \eta Q^* \tag{56}$$

$$\lambda_i^* = p^* + \beta x_i^* \tag{57}$$

Lemma 8. *If* $\Pi_i\left(x_i, x_{-i}^* | q_0\right) > -\infty$ *then*

$$\int_0^\infty e^{-rt} q_i^2 dt < \infty \tag{58}$$

$$\int_0^\infty e^{-rt} x_i^2 dt < \infty \tag{59}$$

Proof. Let $X_{-i}^* = \sum_{j \neq i} x_j^*$ and $Q_{-i}^* = \sum_{j \neq i} q_j^*$. Let $z_{-i} = \gamma - \eta Q_{-i}^*$ and $p_{-i} = \alpha + \beta X_{-i}^*$. If $\Pi_i\left(x_i, x_{-i}^* | q_0\right) > -\infty$ then by the definition of Π_i

$$\begin{aligned}\Pi_i\left(x_i, x_{-i}^* | q_0\right) &= \int_0^\infty e^{-rt} \pi_i dt = \int_0^\infty e^{-rt} \left[z q_i - p x_i\right] dt \\ &= \int_0^\infty e^{-rt} \left[\left(z_{-i} - \eta q_i\right) q_i - \left(p_{-i} + \beta x_i\right) x_i\right] dt \\ &= \int_0^\infty e^{-rt} \left[z_{-i} q_i - \eta q_i^2 - p_{-i} x_i - \beta x_i^2\right] dt \end{aligned} \tag{60}$$

Applying Young's inequality pairwise produces

$$\Pi_i\left(x_i, x_{-i}^* | q_0\right) \leq \int_0^\infty e^{-rt} \left[\frac{1}{2\eta} z_{-i}^2 - \frac{\eta}{2} q_i^2 + \frac{1}{2\beta} p_{-i}^2 - \frac{\beta}{2} x_i^2\right] dt \tag{61}$$

Since z_{-i} and p_{-i} are bounded, there exists $C \in \mathbb{R}$ such that

$$\Pi_i\left(x_i, x_{-i}^* | q_0\right) \leq C - \int_0^\infty e^{-rt} \left[\frac{\eta}{2} q_i^2 + \frac{\beta}{2} x_i^2\right] dt \tag{62}$$

Since $\Pi_i\left(x_i, x_{-i}^* | q_0\right) > -\infty$, we must have (58) and (59). □

Lemma 9. *If* $\Pi_i\left(x_i, x_{-i}^* | q_0\right) > -\infty$ *then* $\lim_{t \to \infty} e^{-rt} \lambda_i^* \left[q_i^* - q_i\right] = 0$

Proof. Since λ_i^* and q_i^* are bounded, $e^{-rt} \lambda_i^* q_i^* \to 0$. Let $g = e^{-rt} q_i$. If $\Pi_i\left(x_i, x_{-i}^* | q_0\right) > -\infty$ then Lemma 8 implies

$$\int_0^\infty g^2 dt = \int_0^\infty e^{-2rt} q_i^2 dt \leq \int_0^\infty e^{-rt} q_i^2 dt < \infty \tag{63}$$

Differentiating g with respect to t obtains

$$\dot{g} = e^{-rt}\dot{q}_i - re^{-rt}q_i = e^{-rt}\left[\dot{q}_i - rq_i\right] = e^{-rt}\left[x_i - (r+\delta)\,q_i\right] \tag{64}$$

Since $(a-b)^2 \leq 2a^2 + 2b^2$

$$\dot{g}^2 \leq 2e^{-2rt}\left[x_i^2 + (r+\delta)^2 q_i^2\right] \leq 2e^{-rt}\left[x_i^2 + (r+\delta)^2 q_i^2\right] \tag{65}$$

Since $\Pi_i\left(x_i, x_{-i}^*|q_0\right) > -\infty$, Lemma 8 implies

$$\int_0^\infty \dot{g}^2 dt \leq 2\int_0^\infty e^{-rt}\left[x_i^2 + (r+\delta)^2 q_i^2\right] dt < \infty \tag{66}$$

Since g and $\dot{g}$ are both square integrable, $g\dot{g}$ is integrable by Cauchy-Schwarz. By the chain rule, $\frac{d}{dt}\left[g^2\right] = 2g\dot{g}$, so by the fundamental theorem of calculus

$$g(t)^2 = g(0)^2 + 2\int_0^t g(s)\,\dot{g}(s)\,ds \tag{67}$$

Since $g\dot{g}$ is integrable, this converges to a finite limit. Since g is square integrable, that limit is zero, so $g^2 \to 0$ and $g \to 0$. Since λ_i^* is bounded, $e^{-rt}\lambda_i^* q_i = \lambda_i^* g \to 0$. Since $e^{-rt}\lambda_i^* q_i^* \to 0$, this implies $e^{-rt}\lambda_i^*\left[q_i^* - q_i\right] \to 0$. □

Lemma 10. $\dot{\lambda}_i^* = (r+\delta)\,\lambda_i^* - z^* + \eta q_i^*$

Proof. As shown in the proof of Lemma 6, x_i^* and q_i^* satisfy the differential equation

$$\beta\dot{x}_i^* = (r+\delta)\left(\frac{\alpha}{n+1} + \beta x_i^*\right) - \frac{\gamma}{n+1} + \eta q_i^* \tag{68}$$

Summing over $i \in N$ produces

$$\beta\dot{X}^* = (r+\delta)\left(\frac{n\alpha}{n+1} + \beta X^*\right) - \frac{n\gamma}{n+1} + \eta Q^* \tag{69}$$

Adding (68) and (69) yields

$$\beta\dot{X}^* + \beta\dot{x}_i^* = (r+\delta)\left(\alpha + \beta X^* + \beta x_i^*\right) - \gamma + \eta Q^* + \eta q_i^* \tag{70}$$

Since $p^* = \alpha + \beta X^*$ and $z^* = \gamma - \eta Q^*$

$$\dot{p}^* + \beta\dot{x}_i^* = (r+\delta)\left(p^* + \beta x_i^*\right) - z^* + \eta q_i^* \tag{71}$$

Since $\lambda_i^* = p^* + \beta x_i^*$

$$\dot{\lambda}_i^* = (r+\delta)\,\lambda_i^* - z^* + \eta q_i^* \tag{72}$$

□

Proof of Theorem 1. Fix $i \in N$. Let $H_i = \pi_i + \lambda_i\left(x_i - \delta q_i\right)$. Since $\partial\pi_i/\partial q_i = z - \eta q_i$, λ_i^* satisfies the adjoint equation $\dot{\lambda}_i = r\lambda_i - \partial H_i/\partial q_i = (r+\delta)\,\lambda_i - \partial\pi_i/\partial q_i$ by Lemma 10. The control x_i is unconstrained and the Hamiltonian is strictly concave in (x_i, q_i) by Lemma 1, so the maximum condition reduces to the interior first order condition $\partial H_i/\partial x_i = 0$. Using $\partial\pi_i/\partial x_i = -p - \beta x_i$ gives $\lambda_i = p + \beta x_i$, so x_i^* maximizes H_i for $\lambda_i =$

λ_i^*. If $\Pi_i\left(x_i, x_{-i}^*|q_0\right) > -\infty$ then $e^{-rt}\lambda_i^*\left[q_i^* - q_i\right] \to 0$ by Lemma 9. Hence x_i^* maximizes $\Pi_i\left(x_i, x_{-i}^*|q_0\right)$ by the Sufficiency Theorem of Mangasarian (1966) and Seierstad and Sydsæter (1977). Since i was arbitrary, (x^*, q_0) is an open loop equilibrium. By Lemma 7, this is the unique open loop equilibrium. □

Proof of Theorem 2. If $x_i = \phi_i(q) = \delta q_i + \kappa(q_i - \bar{q})$ for every investor $i \in N$ and every endowment profile $q \in \mathbb{R}^n$, then $\dot{q}_i = x_i - \delta q_i = \kappa(q_i - \bar{q})$ and $q_i = \bar{q} + (q_{0i} - \bar{q})e^{\kappa t} = q_i^*$, so $x_i = \delta\bar{q} + (\delta + \kappa)(q_{0i} - \bar{q})e^{\kappa t} = x_i^*$. Hence $\Pi_i\left(x_i, x_{-i}^*|q_0\right) = \Pi_i\left(x_i, \phi_{-i}|q_0\right)$ since $\phi_{-i}(q)$ is independent of q_i. By Theorem 1, x_i^* maximizes $\Pi_i\left(x_i, x_{-i}^*|q_0\right)$, so ϕ is a Markov perfect equilibrium. □

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