Subjective Beliefs and Price Dynamics

When is Rationality Enough?

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Abstract

This paper considers asset markets populated by agents who aim to maximize the expected present value of their portfolio. Rational asset valuations are shown to be convex combinations between fundamental values and subjective expectations. The optimal investment strategy consists of buying when the rational asset valuation exceeds the price and selling when the price exceeds the rational asset valuation. Misalignment between between interest rates and subjective discount rates is shown to produce rational asset valuations that deviate from the fundamental value, leading to bubbles and crashes in asset prices.

1 Introduction

This paper considers asset markets populated by agents who aim to maximize the expected present value of their portfolio. In such markets, rational asset valuations are shown to be convex combinations between fundamental values and subjective expectations of future prices. Optimal investment strategies are shown to consist of purchasing assets whenever the price is less than the rational valuation and selling assets whenever the price is greater than the rational valuation.

Conventional notions of equilibrium often rely the assumption that subjective beliefs are consistent with the strategies employed by others.[1] In contrast, the present paper maintains only the weaker assumption that subjective beliefs are consistent with finite order knowledge of rationality.[2] Finite order knowledge of rationality shown to be sufficient for equilibrium in markets with bounded transaction rates if the interest rate coincides with the subjective discount rate.

If the interest rate differs from the subjective discount rate then finite order knowledge of rationality is shown to be insufficient for equilibrium. Misalignment between interest rates and subjective discount rates can produce bubbles and crashes where asset prices temporarily diverge from fundamental values. The larger the misalignment between interest rates and subjective discount rates, the larger the resulting bubbles and crashes.

The remainder of this paper is organized as follows. Section 2 discusses the related literature. Section 3 describes the markets under consideration. Section 4 considers investment strategies. Section 5 considers beliefs structures. Section 6 characterizes the equilibrium price path. Section 7 discusses the role of rationality and section 8 concludes. All proofs are provided in the appendix.

[1] Kreps and Wilson (1982) and Fudenberg and Tirole (1991) provide widely employed equilibrium conditions requiring consistency between strategies and subjective beliefs.

[2] The limited depth of reasoning that results from finite order knowledge of rationality is consistent with the empirical results of Kübler and Weizsäcker (2004) among others.

2 Related Literature

Previous authors have considered the rationalization of price bubbles by the expectation of capital gains in an overlapping generations framework. Tirole (1985) describes perpetually growing asset price bubbles that never burst. Weil (1987) and Martin and Ventura (2012) consider stochastic bubbles of finite duration that eventually burst. Gali (2014) investigates the role of monetary policy in the development of such bubbles.

Others investigate how price bubbles can be rationalized when agents have infinite investment horizons. Kocherlakota (2008) considers rational price bubbles in markets where agents face strict solvency constraints. Hellwig and Lorenzoni (2009) describe rational bubbles in markets where default is penalized by the inability to borrow in the future. Miao and P. Wang (2018) consider rational bubbles in markets where default is penalized by the threat of asset seizure.

Bhattacharyya and Lipman (1995) describe markets where each agent knows only their own personal credit constraint or their own personal investment horizon. The presence of this private information prevents agents from knowing exactly when a bubble will burst. Kyle, Obizhaeva, and Y. Wang (2018) consider bubbles that are generated by agents who exhibit overconfidence in their private information about the growth rate of dividends.

In contrast, the present paper considers fully rational agents who lack common knowledge of rationality but possess some finite order knowledge of rationality. If the interest rate coincides with the subjective discount rate, then knowledge of rationality is sufficient for equilibrium. However, if the interest rate differs from the subjective discount rate, then knowledge of rationality is insufficient for equilibrium. Larger differences between interest rates and subjective discount rates are shown to produce larger deviations from the equilibrium price path.

3 Asset Markets

Consider an asset market populated by a continuum of agents $i \in [0,1]$. Let $p(t) \in \mathbb{R}_+$ denote the price of the asset at time $t \in \mathbb{R}_+$. Let $q_i(t) \in \mathbb{R}$ denote the quantity of assets held by agent i at time t. Let $m_i(t) \in \mathbb{R}$ denote the quantity of money held by agent i at time t. Agent i's wealth $w_i(t)$ is the net value of her portfolio at time t.

$$w_i(t) = m_i(t) + p(t)\, q_i(t) \tag{1}$$

Here $p(t)\, q_i(t)$ denotes the value of the assets held by agent i at time t. Let $r \in \mathbb{R}_{++}$ denote the interest rate and let $z \in \mathbb{R}_{++}$ denote the net rental rate such that agent i's net revenue at time t is given by

$$\dot{m}_i(t) = rm_i(t) + zq_i(t) - p(t)\, \dot{q}_i(t) \tag{2}$$

Here $\dot{m}_i(t)$ denotes the right-derivative[3] of m_i with respect to t, $rm_i(t)$ denotes agent i's interest income, $zq_i(t)$ denotes her rental income, and $p(t)\, \dot{q}_i(t)$ denotes the cost of her asset purchases. Hence the net rate of change in the value of agent i's portfolio at time t is given by

$$\dot{w}_i(t) = rm_i(t) + zq_i(t) + \dot{p}(t)\, q_i(t) \tag{3}$$

Here $\dot{p}(t)\, q_i(t)$ denotes agent i's rate of capital gains at time t. Let $\bar{p} \in \mathbb{R}$ denote the present value of all future cash flows generated by an asset such that

$$\bar{p} = \int_0^\infty e^{-rt} z\, dt = \frac{z}{r} \tag{4}$$

Let T denote a stochastic investment horizon[4] such that $\Pr(T \leq t) = 1 - e^{-t/\beta}$,

[3] In general, we write $\dot{x}(t)$ for the right-derivative of x with respect to t.

[4] The stochastic terminal time T serves as a way of formalizing time preferences. In

so the expected investment horizon is $\beta = E\{T\}$. The expected present value of agent i's portfolio at time T is

$$E\left\{e^{-rT} w_i\left(T\right)\right\} = \frac{1}{\beta} \int_0^\infty e^{-\delta t} w_i\left(t\right) dt \tag{5}$$

Here $\delta = r + 1/\beta$ denotes the subjective discount rate. Let $\zeta\left(t|p\right)$ denote the economic rental rate at time t under the price path p such that

$$\zeta\left(t|p\right) = z + \dot{p}\left(t\right) - rp\left(t\right) \tag{6}$$

The economic rental rate is equal to the accounting rental rate z plus the appreciation rate $\dot{p}\left(t\right)$ less the opportunity cost $rp\left(t\right)$. Proposition 1 states that the expected present value of an agent's portfolio at time T is equal to her initial wealth plus her total discounted economic rent.

Proposition 1. *The expected present value of agents i's wealth at time T under the price path p is given by*

$$E\left\{e^{-rT} w_i\left(T\right)\right\} = w_i\left(0\right) + \int_0^\infty e^{-\delta s} \zeta\left(t|p\right) q_i\left(t\right) dt \tag{7}$$

4 Investment

Let P denote the set of continuous and right-differentiable price paths $p : \mathbb{R}_+ \to \mathbb{R}_+$. Let $p_t : [0, t] \to \mathbb{R}_+$ denote the restriction of the price path $p \in P$ to the closed interval $[0, t]$. Let $\mathcal{H}$ denote the set of price histories such that

$$\mathcal{H} = \{p_t : p \in P, t \in \mathbb{R}_+\} \tag{8}$$

theory, the entire market ends at time T. It would be equivalent to assume that each agent has an independent and identically distributed terminal time T_i and that their assets are inherited by a new agent at this time.

Let $\gamma \in \mathbb{R}_{++}$ denote a finite but arbitrarily large upper bound on transaction rates. Let $\mathcal{D} = [-\gamma, \gamma]$ denote the set of feasible purchase rates. Agent i's investment strategy $\phi_i : \mathcal{H} \to \mathcal{D}$ specifies her net purchase rate as a function of the history p_t such that agent i's asset holdings at time t under the price path p are given by

$$q_i(t|\phi_i, p) = q_i(0) + \int_0^t \phi_i(p_s)\, ds \tag{9}$$

If $q_i(t|\phi_i, p)$ is right-differentiable in t then the right-derivative $\dot{q}_i(t|\phi_i, p) = \phi_i(p_t)$ for almost[5] all $t \in \mathbb{R}_+$. Let $\pi_i(\phi_i|p)$ denote agent i's expected discounted profit under the price path p such that

$$\pi_i(\phi_i|p) = \int_0^\infty e^{-\delta t} \zeta(t|p)\, q_i(t|\phi_i, p)\, dt \tag{10}$$

Let Φ_i denote the set of investment strategies $\phi_i : \mathcal{H} \to \mathcal{D}$ such that $\dot{q}_i(t|\phi_i, p)$ is right-continuous in t for all $p \in P$. Let $\Phi = \prod_{i\in[0,1]} \Phi_i$ denote the set of investment strategy profiles. An investment strategy $\phi_i \in \Phi_i$ is said to be optimal under the price path $p \in P$ if it maximizes agent i's total discounted economic profits under p. Let $\Phi_i^*(p)$ denote the set of agent i's optimal investment strategies under the price path $p \in P$ such that

$$\Phi_i^*(p) = \underset{\phi_i \in \Phi_i}{\operatorname{argmax}}\, \pi_i(\phi_i, p) \tag{11}$$

Let $\lambda(t|p)$ denote the discounted future economic rent on an asset at time $t \in \mathbb{R}_+$ under the price path $p \in P$ such that

$$\lambda(t|p) = \int_t^\infty e^{-\delta(s-t)} \zeta(s|p)\, ds \tag{12}$$

[5] That is to say, for all $t \in \mathbb{R}_+$ save for at most a set of measure zero.

Let α denote the ratio between the interest rate and the subjective discount rate such that $\alpha = \frac{r}{\delta} = \frac{r}{r+1/\beta}$. This ratio is increasing in the interest rate r. It approaches 0 for very small values of r and it approaches 1 for very large values of r. Let τ denote a stochastic investment horizon such that $P(\tau > s) = 1 - e^{-\delta s}$. Proposition 2 characterizes the discounted future economic rent on an asset

Proposition 2. *The discounted future economic rent on an asset under the price path $p \in P$ is given by $\lambda(t|p) = V(t|p) - p(t)$ where*

$$V(t|p) = \alpha\bar{p} + (1-\alpha) E_t \{p(\tau)\}$$

Proposition 2 states that the discounted future economic rent on an asset is given by the difference between $V(t|p)$ and the current asset price. Here $V(t|p)$ is a convex combination between the fundamental value and expected future prices. It equals the fundamental value in the limiting case where α approaches 1. Conversely, the fundamental value has no influence on $V(t|p)$ in the limiting case where α approaches 0.

Proposition 3 characterizes the set of optimal investment strategies. It states that $V(t|p)$ is the rational asset valuation. Optimal investment strategies consist of purchasing assets whenever the rational asset valuation is greater than the current asset price and selling assets whenever the rational asset valuation is less than the current asset price.

Proposition 3. *An investment strategy $\phi_i \in \Phi_i$ is optimal under the price path $p \in P$ if and only if for almost all $t \in \mathbb{R}_+$ we have $\phi_i(p_t) = \gamma$ if $V(t|p) > p(t)$ and $\phi_i(p_t) = -\gamma$ if $V(t|p) < p(t)$.*

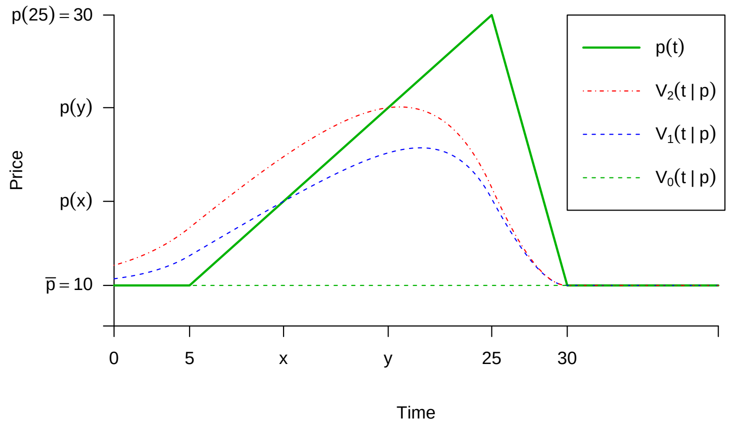

Figure 1: Rational asset valuations for fundamental value $\bar{p} = 10$ and expected investment horizon $\beta = 5$. If $\alpha = 1$ then the rational asset valuation is given by $V_0 (t|p)$ so the optimal investment strategy involves purchasing assets until time $t = 5$. If $\alpha = 1/3$ then the rational asset valuation is given by $V_1 (t|p)$ so the optimal investment strategy involves purchasing assets until time x. If $\alpha = 0.05$ then the rational asset valuation is given by $V_2 (t|p)$ so the optimal investment strategy involves purchasing assets until time y.

5 Beliefs

Let p_t^+ denote the set of price paths that are consistent with the price path p up to time t such that

$$p_t^+ = \{\rho \in P : \rho_t = p_t\} \tag{13}$$

Let ϕ_{it}^+ denote the set of investment strategies that agree with ϕ_i up to time t such that

$$\phi_{it}^+ = \{\varphi_i \in \Phi_i : \varphi_i (p_s) = \phi_i (p_s) \text{ for all } s \in [0, t] \text{ and } p \in P\} \tag{14}$$

Let μ_i denote agent i's belief structure such that $\mu_i (B|p_t)$ is agent i's subjective probability of the event $p \in B$ conditional on the event $p \in p_t^+$ for all B in a sigma algebra over P. Let M_i denote the set of such belief structures. Agent i's expected payoff from the investment strategy ϕ_i under the belief structure

μ_i conditional on the price history $p_t \in \mathcal{H}$ is given by

$$\bar{\pi}_i(\phi_i|\mu_i, p_t) = \int \pi_i(\phi_i|\rho)\, d\mu_i(\rho|p_t) \tag{15}$$

An investment strategy $\phi_i \in \Phi_i$ is said to be optimal under the belief structure $\mu_i \in M_i$ conditional on the price history $p_t \in \mathcal{H}$ if it maximizes agent i's subjective expected payoff under μ_i conditional on p_t. Let $\Phi_i^*(\mu_i|p_t)$ denote the set of agent i's optimal investment strategies under the belief structure $\mu_i \in M_i$ conditional on the price history $p_t \in \mathcal{H}$ such that

$$\Phi_i^*(\mu_i|p_t) = \bigcup_{\phi_i \in \Phi_i} \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \bar{\pi}_i(\varphi_i|\mu_i, p_t) \tag{16}$$

An investment strategy $\phi_i \in \Phi_i$ is said to be optimal under the belief structure $\mu_i \in M_i$ if it is optimal conditional on every price history $p_t \in \mathcal{H}$. Let $\Phi_i^*(\mu_i)$ denote the set of optimal investment strategies under the belief structure $\mu_i \in M_i$ such that

$$\Phi_i^*(\mu_i) = \bigcap_{p_t \in \mathcal{H}} \Phi_i^*(\mu_i|p_t) \tag{17}$$

Let $V(p_t|\mu_i)$ denote the expectation of the rational asset valuation $V(t|p)$ under the price history p_t and the belief structure $\mu_i \in M_i$ such that

$$V(p_t|\mu_i) = \int V(t|\rho)\, d\mu_i(\rho|p_t) \tag{18}$$

Proposition 4 characterizes the set of optimal investment strategies under a given belief structure. It states that optimal investment strategies under the belief structure μ_i involve buying assets when $V(p_t|\mu_i)$ is greater than the current asset price and selling assets when $V(p_t|\mu_i)$ is less than the current asset price. Hence $V(p_t|\mu_i)$ is the rational asset valuation under the price history p_t and the belief structure μ_i.

Proposition 4. *An investment strategy $\phi_i \in \Phi_i$ is optimal under the belief system $\mu_i \in M_i$ if and only if for every price path $p \in P$ and almost all $t \in \mathbb{R}_+$*

$$\phi_i(p_t) = \gamma \quad \text{if } V(p_t|\mu_i) > p(t)$$
$$\phi_i(p_t) = -\gamma \quad \text{if } V(p_t|\mu_i) < p(t)$$

6 Equilibrium

Let $Q(t|\phi, p)$ denote the total quantity of assets demanded by investors at time t under the investment strategy profile $\phi \in \Phi$ and the price path $p \in P$ such that

$$Q(t|\phi, p) = \int_0^1 q_i(t|\phi_i, p)\, di \tag{19}$$

The inverse supply function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing continuously differentiable bijection with bounded elasticity. Let p_ϕ denote the price path generated by the investment strategy profile $\phi \in \Phi$ under such that

$$p_\phi(t) = f(Q(t|\phi, p_\phi)) \tag{20}$$

Since there are a continuum of investors, the price path p_ϕ is fully determined by the opponent strategy profile $\phi_{-i} \in \Phi_{-i}$. An investment strategy $\phi_i \in \Phi_i$ is said to be a best response to the opponent strategy profile $\phi_{-i} \in \Phi_{-i}$ if it maximizes agent i's payoff given ϕ_{-i}. Let $\Phi_i^* : \Phi_{-i} \rightrightarrows \Phi_i$ denote agent i's best response correspondence such that

$$\Phi_i^*(\phi_{-i}) = \operatorname*{argmax}_{\phi_i \in \Phi_i} \pi_i(\phi_i|p_\phi) \tag{21}$$

An investment strategy profile $\phi \in \Phi$ is said to be a Nash equilibrium if $\phi_i \in \Phi_i^*(\phi_{-i})$ for all $i \in [0, 1]$. Proposition 5 establishes the existence of an equilibrium investment strategy profile.

Proposition 5. *There exists an equilibrium strategy profile* $\phi^* \in \Phi$ *such that*

$$\phi_i^* (p_t) = \gamma \operatorname{sgn} (\bar{p} - p(t)) \tag{22}$$

A price path $p \in P$ is said to be an equilibrium price path if it is generated by an equilibrium investment strategy profile ϕ. Proposition 6 characterizes the unique equilibrium price path.

Proposition 6. *There is a unique equilibrium price path* p^* *such that*

$$p^*(t) = \begin{cases} f(Q(0) + \operatorname{sgn}(\bar{p} - p(0))\gamma t) & \text{if } t < t^* \\ \bar{p} & \text{if } t \geq t^* \end{cases} \tag{23}$$

$$t^* = \gamma^{-1} \left| f^{-1}(\bar{p}) - f^{-1}(p(0)) \right| \tag{24}$$

The equilibrium price path depends on the fundamental value $\bar{p}$ and the transaction rate γ. In contrast, it is independent of the expected investment horizon β and the subjective discount rate δ.

7 Rationality

A price path $p \in P$ is said to be feasible if it is generated by an investment strategy profile $\phi \in \Phi$. Let P_f denote the set of feasible price paths such that

$$P_f = \{p_\phi : \phi \in \Phi\} \tag{25}$$

A price history $p_t \in \mathcal{H}$ is said to be feasible if it is the restriction of a feasible price path. Let $\mathcal{H}_f$ denote the set of feasible price histories such that

$$\mathcal{H}_f = \{p_t : p \in P_f, t \in \mathbb{R}_+\} \tag{26}$$

Let p_t^f denote the set of feasible price paths consistent with the price history

$p_t \in \mathcal{H}$ such that

$$p_t^f = \{\rho \in P_f : \rho_t = p_t\} \tag{27}$$

A belief structure $\mu_i \in M_i$ is said to be rational if it is consistent with the set of feasible price paths such that $\mu_i(p_t^f|p_t) = 1$ for all $p_t \in \mathcal{H}_f$. Let M_i^0 denote the set of rational belief structures. An investment strategy is said to be rational if it is optimal under rational beliefs. That is to say, $\phi_i \in \Phi_i$ is a rational strategy if $\phi_i \in \Phi_i^* (\mu_i)$ for some $\mu_i \in M_i^0$. Let Φ_i^0 denote the set of rational strategies such that

$$\Phi_i^0 = \bigcup_{\mu_i \in M_i^0} \Phi_i^* (\mu_i) \tag{28}$$

A price path $p \in P$ is said to be rational if it is generated by a rational investment strategy profile such that $p = p_\phi$ for some $\phi \in \Phi^0$. Let P_0 denote the set of rational price paths such that

$$P_0 = \left\{p_\phi : \phi \in \Phi^R\right\} \tag{29}$$

Let $\mathcal{H}_0$ denote the set of rational price histories such that

$$\mathcal{H}_0 = \{p_t : p \in P_0, t \in \mathbb{R}_+\} \tag{30}$$

Let p_t^0 denote the set of rational price paths that are consistent with the price history $p_t \in \mathcal{H}_0$ such that

$$p_t^0 = \{\rho \in P_0 : \rho_t = p_t\} \tag{31}$$

Proposition 7 states that rationality is sufficient for the equilibrium price path if the interest rate coincides with the subjective discount rate.

Proposition 7. *If* $r = \delta$ *then* $P_0 = \{p^*\}$.

Proposition 8 states that rationality is insufficient for the equilibrium price path if the interest rate differs from the subjective discount rate. Even if all agents are rational, rational price paths may diverge from the equilibrium price path if the interest rate differs from the subjective discount rate.

Proposition 8. *If $r \neq \delta$ then there exists $p \in P_0$ such that $p \neq p^*$.*

Consider the following recursive definitions for n^{th} order knowledge of rationality. Let $\mathcal{H}_n$ denote the set of price histories for price paths $p \in P_n$ such that

$$\mathcal{H}_n = \{p_t : p \in P_n, t \in \mathbb{R}_+\} \tag{32}$$

Let p_t^n denote the set of price paths $\rho \in P_n$ that are consistent with the price history $p_t \in \mathcal{H}_n$ such that

$$p_t^n = \{\rho \in P_n : \rho_t = p_t\} \tag{33}$$

A belief structure $\mu_i \in M_i$ is said to satisfy n^{th} order knowledge of rationality if $\mu_i(p_t^{n-1}|p_t) = 1$ for all $p_t \in \mathcal{H}_{n-1}$. Let M_i^n denote the set of belief structures that satisfy n^{th} order knowledge of rationality. An investment strategy $\phi_i \in \Phi_i$ is said to satisfy n^{th} order knowledge of rationality if it is optimal under a belief structure that satisfies n^{th} order knowledge of rationality. Let Φ_i^n denote the set of agent i's investment strategies that satisfy n^{th} order knowledge of rationality such that

$$\Phi_i^n = \bigcup_{\mu_i \in M_i^n} \Phi_i^* (\mu_i) \tag{34}$$

A price path $p \in P$ is said to satisfy n^{th} order knowledge of rationality if it is generated by an investment strategy profile $\phi \in \Phi^n$. Let P_n denote the set of price paths that satisfy n^{th} order knowledge of rationality such that

$$P_n = \{p_\phi : \phi \in \Phi^n\} \tag{35}$$

Proposition 9 states that finite order knowledge of rationality is insufficient for the equilibrium price path if the interest rate differs from the subjective discount rate. In such cases, prices may diverge from equilibrium even if every agent has n^{th} order knowledge of rationality.

Proposition 9. *If $r \neq \delta$ then for all $n \in \mathbb{N}$ there exists a price path $p \in P_n$ that diverges from equilibrium such that $p \neq p^*$.*

8 Discussion

Rational agents are shown value an asset at a convex combinations between its fundamental value and their subjective expectations about its future prices. The weight placed on fundamentals is shown equal to the ratio between the interest rate and the subjective discount rate. If the interest rate coincides with the subjective discount rate, then the rational asset valuation coincides with the fundamental value. Optimal investment strategies consist of purchasing assets whenever the price is less than the rational valuation and selling assets whenever the price is greater than the rational valuation.

In equilibrium, prices never diverge from fundamental values. Rationality is shown to be sufficient for equilibrium if interest rates coincide with subjective discount rates. Yet even finite order knowledge of rationality is insufficient for equilibrium if interest rates differ from subjective discount rates. Even if every agent has finite order knowledge of rationality, prices may exhibit bubbles and crashes that diverge from fundamental values. The larger the misalignment between interest rates and subjective discount rates, the larger the resulting bubbles and crashes. Further research is needed to empirically test these predictions and generalize the results to markets with production, consumption, and depreciation of multiple distinct asset classes.

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A Proofs

Proof of proposition 1. Since $w_i(t) = m_i(t) + p(t)\, q_i(t)$ we have

$$\dot{w}_i(t) = \dot{m}_i(t) + \dot{p}(t)\, q_i(t) + p(t)\, \dot{q}_i(t)$$

Since $\dot{m}_i(t) = rm_i(t) + zq_i(t) - p(t)\, \dot{q}_i(t)$ we have

$$\begin{aligned}
\dot{w}_i(t) &= rm_i(t) + zq_i(t) + \dot{p}(t)\, q_i(t) \\
&= r\left[w_i(t) - p(t)\, q_i(t)\right] + zq_i(t) + \dot{p}(t)\, q_i(t) \\
&= rw_i(t) + \left[z + \dot{p}(t) - rp(t)\right] q_i(t) \\
&= rw_i(t) + \zeta(t|p)\, q_i(t) \\
w_i(t) &= e^{rt} w_i(0) + e^{rt} \int_0^t e^{-rs} \zeta(s|p)\, q_i(s)\, ds
\end{aligned}$$

Then the expected present value of agent i's wealth at time T is given by

$$\begin{aligned}
E\left\{e^{-rT} w_i(T)\right\} &= w_i(0) + E\left\{\int_0^T e^{-rs} \zeta(s|p)\, q_i(s)\, ds\right\} \\
&= w_i(0) + \frac{1}{\beta} \int_{t=0}^{t=\infty} e^{-t/\beta} \int_{s=0}^{s=t} e^{-rs} \zeta(s|p)\, q_i(s)\, ds\, dt \\
&= w_i(0) + \frac{1}{\beta} \int_{s=0}^{s=\infty} e^{-rs} \zeta(s|p)\, q_i(s) \int_{t=s}^{t=\infty} e^{-t/\beta} dt\, ds \\
&= w_i(0) + \frac{1}{\beta} \int_{s=0}^{s=\infty} e^{-rs} \zeta(s|p)\, q_i(s) \left[\beta e^{-s/\beta}\right] ds \\
&= w_i(0) + \int_0^\infty e^{-\delta s} \zeta(s|p)\, q_i(s)\, ds
\end{aligned}$$

□

Lemma 1. $\pi_i(\phi_i|p) = q_i(0)\,\lambda(0|p) + u_i(\phi_i|p)$ *where*

$$u_i(\phi_i|p) = \int_0^\infty e^{-\delta t}\dot{q}_i(t|\phi_i,p)\,\lambda(t|p)\ dt$$

Proof. Agent i's total discounted economic profit is given by

$$\begin{aligned}
\pi_i(\phi_i|p) &= \int_0^\infty e^{-\delta s}\zeta(s|p)\,q_i(s)\,ds \\
&= \int_0^\infty e^{-\delta s}\zeta(s|p)\left[q_i(0) + \int_0^s \dot{q}_i(t|\phi_i,p)\,dt\right]ds \\
&= q_i(0)\,\lambda(0|p) + \int_{s=0}^{s=\infty} e^{-\delta s}\zeta(s|p)\int_{t=0}^{t=s}\dot{q}_i(t|\phi_i,p)\,dt\,ds
\end{aligned}$$

The second term can be written as

$$\begin{aligned}
& \int_{t=0}^{t=\infty}\dot{q}_i(t|\phi_i,p)\int_{s=t}^{s=\infty} e^{-\delta s}\zeta(s|p)\,ds\,dt \\
&= \int_{t=0}^{t=\infty} e^{-\delta t}\dot{q}_i(t|\phi_i,p)\,e^{\delta t}\int_{s=t}^{s=\infty} e^{-\delta s}\zeta(s|p)\,ds\,dt \\
&= \int_0^\infty e^{-\delta t}\dot{q}_i(t|\phi_i,p)\,\lambda(t|p)\,dt
\end{aligned}$$

□

Lemma 2. *The expected value of* $\dot{p}(\tau)$ *at time* t *is*

$$E_t\left\{\dot{p}(\tau)\right\} = \delta E\left\{p(\tau)\right\} - \delta p(t)$$

Proof. Since $\Pr(\tau \leq s) = 1 - e^{-\delta s}$ we have

$$\begin{aligned}
E_t\left\{p(\tau)\right\} &= p(t) + E\left\{\int_t^\tau \dot{p}(x)\,dx \;\middle|\; \tau > t\right\} \\
E_t\left\{p(\tau)\right\} &= p(t) + \delta \int_{s=t}^{s=\infty} e^{-\delta(s-t)} \int_{x=t}^{x=s} \dot{p}(x)\,dx\,ds \\
E_t\left\{p(\tau)\right\} &= p(t) + \delta e^{\delta t} \int_{x=t}^{x=\infty} \dot{p}(x) \int_{s=x}^{s=\infty} e^{-\delta s} ds\,dx \\
E_t\left\{p(\tau)\right\} &= p(t) + \delta e^{\delta t} \int_{x=t}^{x=\infty} \dot{p}(x) \left[\frac{e^{-\delta x}}{\delta}\right] dx \\
E_t\left\{p(\tau)\right\} &= p(t) + e^{\delta t} \int_{x=t}^{x=\infty} e^{-\delta x} \dot{p}(x)\; dx \\
E_t\left\{p(\tau)\right\} &= p(t) + \frac{1}{\delta} E_t\left\{\dot{p}(\tau)\right\} \\
E_t\left\{\dot{p}(\tau)\right\} &= \delta E_t\left\{p(\tau)\right\} - \delta p(t)
\end{aligned}$$

□

Proof of proposition 2. By definition (12) the discounted future economic rent on an asset is given by

$$\begin{aligned}
\lambda(t|p) &= \int_t^\infty e^{-\delta(s-t)}\zeta(s|p_\phi)\,ds \\
\delta\lambda(t|p) &= \delta\int_t^\infty e^{-\delta(s-t)}\zeta(s|p_\phi)\,ds = E_t\{\zeta(\tau|p)\} \\
\delta\lambda(t|p) &= E_t\{z+\dot{p}(\tau)-rp(\tau)\} \\
\delta\lambda(t|p) &= z+E_t\{\dot{p}(\tau)\}-rE_t\{p(\tau)\}
\end{aligned}$$

By lemma 2 we have $E_t\{\dot{p}(\tau)\} = \delta E_t\{p(\tau)\} - \delta p(t)$ so

$$\begin{aligned}
\delta\lambda(t|p) &= z+\delta E_t\{p(\tau)\}-\delta p(t)-rE_t\{p(\tau)\} \\
\delta\lambda(t|p) &= z-\delta p(t)+(\delta-r)E_t\{p(\tau)\} \\
\lambda(t|p) &= \frac{z}{\delta}-p(t)+\left(\frac{\delta-r}{\delta}\right)E_t\{p(\tau)\} \\
\lambda(t|p) &= \left(\frac{r}{\delta}\right)\frac{z}{r}-p(t)+\left(\frac{\delta-r}{\delta}\right)E_t\{p(\tau)\} \\
\lambda(t|p) &= \alpha\bar{p}-p(t)+(1-\alpha)E_t\{p(\tau)\} \\
\lambda(t|p) &= V(t|p)-p(t)
\end{aligned}$$

□

Proof of proposition 3. Suppose that $\phi_i \in \Phi_i$ is an optimal investment strategy under the price path $p \in P$. If $\dot{q}_i(t|\phi_i,p) = \gamma$ for $\lambda(t|p) > 0$ and $\dot{q}_i(t|\phi_i,p) = -\gamma$ for $\lambda(t|p) < 0$ then ϕ_i maximizes $\pi_i(\phi_i|p)$ over Φ_i by lemma 1. Conversely, suppose $\dot{q}_i(t|\phi_i,p) \neq \gamma\ \mathrm{sgn}(\lambda(t|p))$ for some $t \in \mathbb{R}_+$ where $\lambda(t|p) \neq 0$. Since $\lambda(t|p)$ is continuous and $\dot{q}_i(t|\phi_i,p)$ is right-continuous, there exists $\varepsilon > 0$ such that $\lambda(s|p) \neq 0$ and $\dot{q}_i(s|\phi_i,p) \neq \gamma\ \mathrm{sgn}(\lambda(t|p))$ for all $s \in (t, t+\varepsilon)$. But then ϕ_i would be suboptimal by lemma 1. Since $\dot{q}_i(t|\phi_i,p) = \phi_i(p_t)$ for almost all $t \in \mathbb{R}_+$ we have $\phi_i(p_t)\,|\mathrm{sgn}(\lambda(t|p))| = \gamma\,\mathrm{sgn}(\lambda(t|p))$ for almost all $t \in \mathbb{R}_+$. □

Proof of proposition 4. By lemma 1, agent i's payoff under $p \in P$ is

$$\pi_i(\phi_i|p) = q_i(0)\lambda(0|p) + \int_0^\infty e^{-\delta t}\dot{q}_i(t|\phi_i, p)\lambda(t|p)\,dt$$

If $\mu_i \in M_i$ then $\phi_i \in \Phi_i^*(\mu_i)$ if and only if for all $p \in P$ and $t \in \mathbb{R}_+$

$$\begin{aligned}
\phi_i &\in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_{p_t^+} \pi_i(\varphi_i|\rho)\,d\mu_i(\rho|p_t) \\
\phi_i &\in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_{p_t^+} \int_0^\infty e^{-\delta s}\dot{q}_i(s|\varphi_i, \rho)\lambda(s|\rho)\,ds\,d\mu_i(\rho|p_t) \\
\phi_i &\in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_0^\infty e^{-\delta s} \int_{p_t^+} \dot{q}_i(s|\varphi_i, \rho)\lambda(s|\rho)\,d\mu_i(\rho|p_t)\,ds
\end{aligned}$$

For almost all $t \in \mathbb{R}_+$ we have $\dot{q}_i(t|\varphi_i, \rho) = \varphi_i(\rho_t)$ so

$$\phi_i \in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_0^\infty e^{-\delta s} \int_{p_t^+} \varphi_i(\rho_t)\lambda(s|\rho)\,d\mu_i(\rho|p_t)\,ds$$

Since $\lambda(s|\rho) = V(t|\rho) - \rho(t)$ we have

$$\phi_i \in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_0^\infty e^{-\delta s} \int_{p_t^+} \varphi_i(\rho_t)\left[V(t|\rho) - \rho(t)\right]d\mu_i(\rho|p_t)\,ds$$

Since $\rho_t = p_t$ for all $\rho \in p_t^+$ we have

$$\begin{aligned}
\phi_i &\in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_0^\infty e^{-\delta s} \int_{p_t^+} \varphi_i(p_t)\left[V(t|\rho) - p(t)\right]d\mu_i(\rho|p_t)\,ds \\
\phi_i &\in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_0^\infty e^{-\delta s}\varphi_i(p_t)\left[\int_{p_t^+} V(t|\rho)\,d\mu_i(\rho|p_t) - p(t)\right]ds \\
\phi_i &\in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_0^\infty e^{-\delta s}\varphi_i(p_t)\left[V(p_t|\mu_i) - p(t)\right]ds
\end{aligned}$$

Hence for almost all $t \in \mathbb{R}_+$ we have

$$\begin{aligned}
\phi_i(p_t) &= \gamma \quad \text{if } V(p_t|\mu_i) > p(t) \\
\phi_i(p_t) &= -\gamma \quad \text{if } V(p_t|\mu_i) < p(t)
\end{aligned}$$

□

Proof of proposition 5. Let $\phi \in \Phi$ be an investment strategy profile such that

$$\phi_i\left(p_t\right) = \gamma \operatorname{sgn}\left(\bar{p} - p\left(t\right)\right)$$

Let $t^* \in \mathbb{R}_+$ denote the time at which p_ϕ reaches $\bar{p}$ such that

$$\gamma t^* = \left|f^{-1}\left(\bar{p}\right) - f^{-1}\left(p\left(0\right)\right)\right|$$

Hence if $t \geq t^*$ then $p_\phi\left(t\right) = \bar{p} = \frac{z}{r}$. Conversely, if $t < t^*$ then

$$\operatorname{sgn}\left(\dot{p}_\phi\left(t\right)\right) = \operatorname{sgn}\left(\bar{p} - p_\phi\left(t\right)\right) = \operatorname{sgn}\left(\bar{p} - p_\phi\left(0\right)\right)$$

By (12) we have

$$\begin{aligned}
\lambda\left(t|p_\phi\right) &= \int_t^\infty e^{-\delta(s-t)}\zeta\left(s|p_\phi\right) ds \\
\lambda\left(t|p_\phi\right) &= \int_t^\infty e^{-\delta(s-t)}\left[z + \dot{p}_\phi\left(t\right) - r p_\phi\left(t\right)\right] ds
\end{aligned}$$

Hence if $t \geq t^*$ then

$$\lambda\left(t|p_\phi\right) = \int_t^\infty e^{-\delta(s-t)}\left[z + 0 - z\right] ds = 0$$

And if $t < t^*$ then

$$\begin{aligned}
\lambda\left(t|p_\phi\right) &= \int_t^{t^*} e^{-\delta(s-t)}\left[z + \dot{p}_\phi\left(t\right) - r p_\phi\left(t\right)\right] ds \\
\lambda\left(t|p_\phi\right) &= \int_t^{t^*} e^{-\delta(s-t)} r\left[\frac{z}{r} + \frac{1}{r}\dot{p}_\phi\left(t\right) - p_\phi\left(t\right)\right] ds \\
\lambda\left(t|p_\phi\right) &= \int_t^{t^*} e^{-\delta(s-t)} r\left[\bar{p} - p_\phi\left(t\right) + \frac{1}{r}\dot{p}_\phi\left(t\right)\right] ds
\end{aligned}$$

Since $\operatorname{sgn}\left(\dot{p}_\phi\left(t\right)\right) = \operatorname{sgn}\left(\bar{p} - p_\phi\left(t\right)\right) = \operatorname{sgn}\left(\bar{p} - p_\phi\left(0\right)\right)$ for $t < t^*$ we have

$$\operatorname{sgn}\left(\lambda\left(t|p_\phi\right)\right) = \operatorname{sgn}\left(\bar{p} - p_\phi\left(t\right)\right)$$

Thus ϕ_i is optimal under p_ϕ by proposition 3. □

Lemma 3. *For every feasible price path $p \in P_f$ we have*

$$E_t\left\{p(\tau)\right\} \leq \int_0^\infty e^{-s} f\left(f^{-1}(p(t)) + \frac{\gamma}{\delta}s\right) ds$$

Proof. If $p \in P_f$ then $p(t) = f(Q_\phi(t))$ for some $\phi \in \Phi$ so

$$p(t+s) \leq f\left(f^{-1}(p(t)) + \gamma s\right)$$

Taking expectation conditional on $\tau \geq t$ yields

$$\begin{aligned} E_t\left\{p(\tau)\right\} &= \delta \int_0^\infty e^{-\delta s} p(t+s)\, ds \\ &\leq \delta \int_0^\infty e^{-\delta s} f\left(f^{-1}(p(t)) + \gamma s\right) ds \\ &\leq \int_0^\infty e^{-u} f\left(f^{-1}(p(t)) + \frac{\gamma}{\delta}u\right) du \qquad \text{where } u = \delta s \end{aligned}$$

□

Lemma 4. *There exists $\eta \in \mathbb{R}_{++}$ such that for all $a, b \in \mathbb{R}_{++}$*

$$\frac{f(b)}{f(a)} < \left(\frac{b}{a}\right)^\eta$$

Proof. Since f is continuously differentiable with bounded elasticity there exists $\eta \in \mathbb{R}_{++}$ such that

$$\begin{aligned} f'(x)\frac{x}{f(x)} &< \eta \qquad \text{for all } x > 0 \\ \frac{f'(x)}{f(x)} &< \frac{\eta}{x} \\ \int_a^b \frac{f'(x)}{f(x)} dx &< \int_a^b \frac{\eta}{x} dx \\ \log(f(b)) - \log(f(a)) &< \eta \log(b) - \eta \log(a) \\ \frac{f(b)}{f(a)} &< \left(\frac{b}{a}\right)^\eta \end{aligned}$$

□

Lemma 5. $\lim_{x\to\infty}\int_0^\infty e^{-s}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x}ds=1$

Proof. Since f is increasing and bijective $\lim_{x\to\infty}f^{-1}(x)=\infty$ and

$$\begin{aligned}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x} &\geq 1 \quad \text{for } s\geq 0\\ \int_0^\infty e^{-s}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x}ds &\geq 1\end{aligned}$$

By lemma 4 we have

$$\begin{aligned}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x} &= \frac{f\left(f^{-1}(x)+\gamma s\right)}{f\left(f^{-1}(x)\right)}\\ \frac{f\left(f^{-1}(x)+\gamma s\right)}{x} &< \left(\frac{f^{-1}(x)+\gamma s}{f^{-1}(x)}\right)^\eta\\ \frac{f\left(f^{-1}(x)+\gamma s\right)}{x} &< \left(1+\frac{\gamma s}{f^{-1}(x)}\right)^\eta\\ \int_0^\infty e^{-s}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x}ds &< \int_0^\infty e^{-s}\left(1+\frac{\gamma s}{f^{-1}(x)}\right)^\eta ds\end{aligned}$$

Since $\lim_{x\to\infty}f^{-1}(x)=\infty$ taking the limit as $x\to\infty$ obtains

$$\lim_{x\to\infty}\int_0^\infty e^{-s}\left(1+\frac{\gamma s}{f^{-1}(x)}\right)^\eta ds = 1$$

□

Lemma 6. *There exists $\hat{p} > \bar{p}$ such that*

$$p(t) > \hat{p} \implies \lambda(t|p) < 0 \quad \text{for all } p \in P_f$$

Proof. By proposition 2 we have

$$\lambda(t|p) = \alpha\bar{p} + (1-\alpha) E_t \{p(\tau)\} - p(t)$$

By lemma 3 we can write

$$\lambda(t|p) \leq \alpha\bar{p} + (1-\alpha) \int_0^\infty e^{-t} f\left(f^{-1}(p(t)) + \frac{\gamma}{\delta}t\right) dt - p(t)$$

Dividing by $p(t)$ obtains

$$\frac{\lambda(t|p)}{p(t)} \leq \alpha\frac{\bar{p}}{p(t)} + (1-\alpha) \int_0^\infty e^{-t} \frac{f\left(f^{-1}(p(t)) + \frac{\gamma}{\delta}t\right)}{p(t)} dt - 1$$

By lemma 5 taking the limit as $p(t) \to \infty$ obtains

$$\lim_{p(t)\to\infty} \alpha\frac{\bar{p}}{p(t)} + (1-\alpha) \int_0^\infty e^{-t} \frac{f\left(f^{-1}(p(t)) + \frac{\gamma}{\delta}t\right)}{p(t)} dt - 1 = -\alpha$$

□

Lemma 7. *There exists $\check{p} \in (0, \bar{p})$ such that*

$$p(t) < \check{p} \implies \lambda(t|p) > 0 \quad \text{for all } p \in P_f$$

Proof. By proposition 2 we have

$$\lambda(t|p) = \alpha\bar{p} + (1-\alpha) E_t \{p(\tau)\} - p(t)$$

Since $p : \mathbb{R}_+ \to \mathbb{R}_{++}$ and $\alpha \in (0, 1]$

$$\lambda(t|p) \geq \alpha\bar{p} - p(t)$$

Hence $\lambda(t|p) > 0$ if $p(t) < \alpha\bar{p} \in \mathbb{R}_{++}$. □

Lemma 8. *If $\phi \in \Phi$ is a Nash equilibrium and $p_\phi(t) > \bar{p}$ then $\dot{Q}_\phi(t) = -\gamma$.*

Proof. By lemma 6 there exists $\hat{p}_0 > \bar{p}$ such that for all $p \in P_f$

$$p(t) > \hat{p}_0 \implies \lambda(t|p) < 0$$

For all $n \in \mathbb{N}$ let $\hat{p}_n \in \mathbb{R}_{++}$ such that

$$\hat{p}_n = \frac{z}{\delta} \sum_{k=0}^{n-1} \left(\frac{\delta - r}{\delta} \right)^k + \left(\frac{\delta - r}{\delta} \right)^n \hat{p}_0$$

Then by proposition 3

$$p_\phi(t) > \hat{p}_0 \implies \dot{Q}_\phi(t) = -\gamma$$

For the inductive hypothesis, suppose

$$p_\phi(t) > \hat{p}_n \implies \dot{Q}_\phi(t) = -\gamma$$

Hence if $p_\phi(t) \leq \hat{p}_n$ then $E_t\{p_\phi(\tau)\} \leq \hat{p}_n$ and by proposition 2

$$\begin{aligned} \lambda(t|p_\phi) &\leq z - \delta p_\phi(t) + (\delta - r)\hat{p}_n \\ \lambda(t|p_\phi) &\leq \delta[\hat{p}_{n+1} - p_\phi(t)] \end{aligned}$$

Then by proposition 3

$$p_\phi(t) > \hat{p}_{n+1} \implies \dot{Q}_\phi(t) = -\gamma$$

So by induction we have

$$p_\phi(t) > \hat{p}_n \implies \dot{Q}_\phi(t) = -\gamma \quad \text{for all } n \in \mathbb{N}$$

Taking the limit as $n \to \infty$ obtains

$$\lim_{n\to\infty} \hat{p}_n = \frac{z}{\delta} \sum_{k=0}^{\infty} \left(\frac{\delta - r}{\delta} \right)^k = \frac{z}{\delta} \left(\frac{\delta}{r} \right) = \frac{z}{r} = \bar{p}$$

□

Lemma 9. *If $\phi \in \Phi$ is a Nash equilibrium and $p_\phi(t) < \bar{p}$ then $\dot{Q}_\phi(t) = \gamma$.*

Proof. By lemma 7 there exists $\check{p}_0 < \bar{p}$ such that for all $p \in P_f$

$$p(t) < \check{p}_0 \implies \lambda(t|p) > 0$$

For all $n \in \mathbb{N}$ let $\check{p}_n \in \mathbb{R}_{++}$ such that

$$\check{p}_n = \frac{z}{\delta} \sum_{k=0}^{n-1} \left(\frac{\delta - r}{\delta} \right)^k + \left(\frac{\delta - r}{\delta} \right)^n \check{p}_0$$

Then by proposition 3

$$p_\phi(t) < \check{p}_0 \implies \dot{Q}_\phi(t) = \gamma$$

For the inductive hypothesis, suppose

$$p_\phi(t) < \check{p}_n \implies \dot{Q}_\phi(t) = \gamma$$

Hence if $p_\phi(t) \geq \check{p}_n$ then $E_t\{p_\phi(\tau)\} \geq \check{p}_n$ and by proposition 2

$$\begin{aligned} \lambda(t|p_\phi) &\geq z - \delta p_\phi(t) + (\delta - r)\check{p}_n \\ \lambda(t|p_\phi) &\geq \delta[\check{p}_{n+1} - p_\phi(t)] \end{aligned}$$

Then by proposition 3

$$p_\phi(t) < \check{p}_{n+1} \implies \dot{Q}_\phi(t) = \gamma$$

So by induction we have

$$p_\phi(t) < \check{p}_n \implies \dot{Q}_\phi(t) = \gamma \quad \text{for all } n \in \mathbb{N}$$

Taking the limit as $n \to \infty$ obtains

$$\lim_{n\to\infty} \check{p}_n = \frac{z}{\delta} \sum_{k=0}^{\infty} \left(\frac{\delta - r}{\delta} \right)^k = \frac{z}{\delta} \left(\frac{\delta}{r} \right) = \frac{z}{r} = \bar{p}$$

□

Lemma 10. *If $\phi \in \Phi$ is a Nash equilibrium and $p_\phi(t) = \bar{p}$ then $\dot{Q}_\phi(t) = 0$.*

Proof. By (20) the price path generated by ϕ satisfies

$$\dot{p}_\phi(s) = f'(Q_\phi(s))\,\dot{Q}(s)$$

Now $\dot{Q}_\phi(t)$ is right-continuous in t since $\dot{q}_i(t|\phi_i, p_\phi)$ is right-continuous in t. Hence if $\dot{Q}(t) > 0$ then there exists $s > t$ such that $\dot{Q}(\ell) > 0$ for $\ell \in [t, s]$. But then $p_\phi(s) > \bar{p}$ and $\dot{p}(s) > 0$, which contradicts lemma 8. Conversely, if $\dot{Q}(t) < 0$ then there exists $s > t$ such that $p_\phi(s) < \bar{p}$ and $p(s) < 0$, which contradicts lemma 9. □

Proof of proposition 6. By proposition 5 there exists a Nash equilibrium investment strategy profile ϕ^* such that $p_{\phi^*}(t) = p^*(t)$ for all $t \in \mathbb{R}_+$. Suppose ϕ is a Nash equilibrium strategy profile. By lemma 8, $\dot{Q}_\phi(t) = -\gamma$ if $p_\phi(t) > \bar{p}$. By lemma 9, $\dot{Q}_\phi(t) = \gamma$ if $p_\phi(t) < \bar{p}$. Let $t^* = \gamma^{-1}\left|f^{-1}(\bar{p}) - Q_\phi(0)\right|$. Since $p_\phi(t) = f(Q_\phi(t))$ we have $Q_\phi(t) = Q(0) + \operatorname{sgn}(\bar{p} - p(0))\,\gamma t$ for all $t < t^*$ and $Q_\phi(t^*) = f^{-1}(\bar{p})$. By lemma 10, $\dot{Q}_\phi(t) = 0$ if $p_\phi(t) = \bar{p}$. Hence $Q_\phi(t) = f^{-1}(\bar{p})$ for all $t \geq t^*$. □

Proof of proposition 7. If $r = \delta$ then by proposition 2 the total discounted economic rent is given by $\lambda(t|p) = \bar{p} - p(t)$. By lemma 1, agent i's payoff under the price path $p \in P$ can be written as

$$\begin{aligned}\pi_i(\phi_i|p) &= q_i(0)\lambda(0|p) + \int_0^\infty e^{-\delta t}\dot{q}_i(t|\phi_i, p)\lambda(t|p)\,dt \\ &= q_i(0)[\bar{p} - p(0)] + \int_0^\infty e^{-\delta t}\dot{q}_i(t|\phi_i, p)[\bar{p} - p(t)]\,dt\end{aligned}$$

If ϕ is a rational strategy profile then ϕ_i is a rational investment strategy and there exists $\mu_i \in M$ such that

$$\begin{aligned}\phi_i &\in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_{p_t^+} \pi_i(\varphi_i|\rho)\,d\mu_i(\rho|p_t) \qquad \text{for all } p_t \in \mathcal{H}_f \\ \phi_i &\in \operatorname*{argmax}_{\varphi_i \in \phi_{it}^+} \int_{p_t^+}\int_0^\infty e^{-\delta s}\dot{q}_i(s|\varphi_i, \rho)[\bar{p} - \rho(s)]\,ds\,d\mu_i(\rho|p_t)\end{aligned}$$

If $\rho \in p_t^+$ then ρ is continuous so there exists $\varepsilon > 0$ such that

$$\begin{aligned}p(t) < \bar{p} &\implies \rho(s) < \bar{p} \quad \text{for all } s \in [t, t+\varepsilon] \\ p(t) > \bar{p} &\implies \rho(s) > \bar{p} \quad \text{for all } s \in [t, t+\varepsilon]\end{aligned}$$

Since $\dot{q}_i(t|\varphi_i, \rho)$ is right continuous in t, optimal investment strategies satisfy

$$\begin{aligned}\dot{q}_i(t|\phi_i, p_t) &= \gamma \quad \text{if } p(t) < \bar{p} \\ \dot{q}_i(t|\phi_i, p_t) &= -\gamma \quad \text{if } p(t) > \bar{p}\end{aligned}$$

Hence the price path p_ϕ must equal the unique equilibrium price path p^* by proposition 6. □

Lemma 11. *If $r < \delta$ then there exists $\varphi \in \Phi^0$ such that*

$$\begin{aligned} \varphi_i(p_t) &= \begin{cases} \gamma & \text{if } t \in [t^*, t^0] \\ \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{otherwise} \end{cases} \\ t_0 > t^* &= \gamma^{-1} \left| f^{-1}(\bar{p}) - f^{-1}(p(0)) \right| \end{aligned}$$

Proof. Let $\phi \in \Phi$ such that for every feasible history $p_t \in \mathcal{H}_f$ we have

$$\phi_i(p_t) = \begin{cases} \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{if } t < t^* \\ \gamma & \text{if } t \geq t^* \end{cases}$$

Then $p_\phi(t^*) = \bar{p}$ and $E_x\{p_\phi(\tau)\} > \bar{p}$. If $r < \delta$ then by proposition 2 the discounted future economic rent at time t^* under the price path p_ϕ satisfies

$$\begin{aligned} \lambda(t^*|p_\phi) &= \alpha\bar{p} + (1-\alpha) E_t\{p_\phi(\tau)\} - p_\phi(t^*) \qquad \text{where } \alpha = \frac{r}{\delta} \in (0,1) \\ &= \alpha\bar{p} + (1-\alpha) E_t\{p_\phi(\tau)\} - \bar{p} \\ &= (1-\alpha)\left[E_t\{p_\phi(\tau)\} - \bar{p}\right] > 0 \end{aligned}$$

Since $\lambda(t|p_\phi)$ is continuous in t there exists $t' > t^*$ such that $\lambda(t|p_\phi) > 0$ for all $t \in [t^*, t']$. Hence by lemma 6 there exists $t_0 > t^*$ such that

$$t_0 = \min\{t \geq t^* : \lambda(t|p_\phi) < 0\}$$

For all $p_t \in \mathcal{H}_f$ let $p_t^\triangleright \in p_t^+$ such that

$$\begin{aligned} p_t^\triangleright(s) &= p_t(s) \quad \text{if } s \in [0,t) \\ \dot{p}_t^\triangleright(s) &= \operatorname{sgn}(\bar{p} - p_t^\triangleright(s))\, \gamma f'\left(f^{-1}(p_t^\triangleright(s))\right) \quad \text{if } s \geq t \end{aligned}$$

Let $\mu_i \in M_i^0$ such that for all $p_t \in \mathcal{H}_f$ we have $\mu_i(p_\phi|p_t) = 1$ if $p_\phi \in p_t^+$ and $\mu_i(p_t^\triangleright|p_t) = 1$ otherwise. Let $\varphi \in \Phi$ such that for all $p_t \in \mathcal{H}_f$

$$\varphi_i(p_t) = \begin{cases} \operatorname{sgn}(\lambda(t|p_\phi))\,\gamma & \text{if } p_\phi \in p_t^+ \\ \operatorname{sgn}(\lambda(t|p_t^\triangleright))\,\gamma & \text{otherwise} \end{cases}$$

Hence $\varphi_i \in \Phi_i^*(\mu_i)$ by proposition 3 so $\varphi_i \in \Phi_i^0$ since $\mu_i \in M_i^0$. $\square$

Lemma 12. *If $r < \delta$ and $\phi \in \Phi^n$ such that*

$$\begin{aligned} \phi_i(p_t) &= \begin{cases} \gamma & \text{if } t \in [t^*, t_n] \\ \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{otherwise} \end{cases} \\ t_n > t^* &= \gamma^{-1} \left| f^{-1}(\bar{p}) - f^{-1}(p(0)) \right| \end{aligned}$$

then there exists $t_{n+1} > t^*$ and $\varphi \in \Phi^{n+1}$ such that

$$\varphi_i(p_t) = \begin{cases} \gamma & \text{if } t \in [t^*, t_{n+1}] \\ \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{otherwise} \end{cases}$$

Proof. By proposition 2 the discounted future economic rent at time t^* under the price path p_ϕ satisfies

$$\begin{aligned} \lambda(t^*|p_\phi) &= \alpha\bar{p} + (1-\alpha) E_t\{p_\phi(\tau)\} - p_\phi(t^*) \qquad \text{where } \alpha = \frac{r}{\delta} \in (0,1) \\ &= \alpha\bar{p} + (1-\alpha) E_t\{p_\phi(\tau)\} - \bar{p} \\ &= (1-\alpha)\left[E_t\{p_\phi(\tau)\} - \bar{p}\right] > 0 \end{aligned}$$

Since $\lambda(t|p_\phi)$ is continuous in t there exists $t' > t^*$ such that $\lambda(t|p_\phi) > 0$ for all $t \in [t^*, t']$. Hence by lemma 6 there exists $t_{n+1} > t^*$ such that

$$t_{n+1} = \min\{t \geq t^* : \lambda(t|p_\phi) < 0\}$$

For all $p_t \in \mathcal{H}_f$ let $p_t^\triangleright \in p_t^+$ such that

$$\begin{aligned} p_t^\triangleright(s) &= p_t(s) \quad \text{if } s \in [0,t] \\ \dot{p}_t^\triangleright(s) &= \operatorname{sgn}(\bar{p} - p_t^\triangleright(s))\,\gamma f'\left(f^{-1}(p_t^\triangleright(s))\right) \quad \text{if } s \geq t \end{aligned}$$

Let $\mu_i \in M_i^{n+1}$ such that for all $p_t \in \mathcal{H}_f$ we have $\mu_i(p^n|p_t) = 1$ if $p^n \in p_t^+$ and $\mu_i(p_t^\triangleright|p_t) = 1$ otherwise. Let $\varphi \in \Phi$ such that for all $p_t \in \mathcal{H}_f$

$$\varphi_i(p_t) = \begin{cases} \operatorname{sgn}(\lambda(t|p^n))\,\gamma & \text{if } p^n \in p_t^+ \\ \operatorname{sgn}(\lambda(t|p_t^\triangleright))\,\gamma & \text{otherwise} \end{cases}$$

Hence $\varphi_i \in \Phi_i^*(\mu_i)$ by proposition 3 so $\varphi_i \in \Phi_i^{n+1}$ since $\mu_i \in M_i^{n+1}$. □

Proof of proposition 8. By proposition 6 the unique equilibrium price path is

$$p^*(t) = \begin{cases} f\left(Q(0) + \operatorname{sgn}(\bar{p} - p(0))\,\gamma t\right) & \text{if } t < t^* \\ \bar{p} & \text{if } t \geq t^* \end{cases}$$

$$t^* = \gamma^{-1}\left|f^{-1}(\bar{p}) - f^{-1}(p(0))\right|$$

By lemma 11 there exists $t_0 > t^*$ and $\varphi \in \Phi^0$ such that

$$\varphi_i(p_t) = \begin{cases} \gamma & \text{if } t \in [t^*, t_0] \\ \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{otherwise} \end{cases}$$

But then $p_\varphi(t_0) > \bar{p} = p^*(t_0)$. □

Proof of proposition 9. By lemma 11 there exists $t_0 > t^*$ and $\phi^0 \in \Phi^0$ with

$$\phi_i^0(t) = \begin{cases} \gamma & \text{if } t \in [t^*, t_0] \\ \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{otherwise} \end{cases}$$

By lemma 12 if $t_k > t^*$ and $\phi^k \in \Phi^n$ such that

$$\phi_i^k(p_t) = \begin{cases} \gamma & \text{if } t \in [t^*, t_k] \\ \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{otherwise} \end{cases}$$

then there exists $t_{k+1} > t^*$ and $\phi^{k+1} \in \Phi^{k+1}$

$$\phi_i^{k+1}(t) = \begin{cases} \gamma & \text{if } t \in [t^*, t_{k+1}] \\ \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{otherwise} \end{cases}$$

Hence for all $n \in \mathbb{N}$ there exists $t_n > t^*$ and $\varphi \in \Phi^n$ such that

$$\varphi_i(t) = \begin{cases} \gamma & \text{if } t \in [t^*, t_n] \\ \gamma \operatorname{sgn}(\bar{p} - p(t)) & \text{otherwise} \end{cases}$$

But then $p_\varphi(t_n) > \bar{p} = p^*(t_n)$ by proposition 6. □