Multi-battle contests over complementary battlefields

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Abstract

This paper studies contests with complementary prizes where each agent simultaneously distributes a fixed budget over multiple battlefields. Each battlefield has a single prize which is divided among the competitors in proportion to an arbitrary power function of investment levels. If objective functions exhibit constant subunitary elasticity of substitution between prize shares, a unique pure strategy Nash equilibrium is shown to exist under which investment levels are proportional to prize values. In contrast, Blotto contests with linear objectives have only mixed strategy Nash equilibria if battlefield success functions are sufficiently sensitive to investment levels. Sufficient complementarity between prize shares allows pure strategy Nash equilibria to exist under arbitrarily sensitive battlefield success functions.

1 Introduction

An agent's value for one resource often depends on their other resources. Military factions compete for both air supremacy and ground supremacy. The marginal increase in control over a contested region from additional air supremacy may depend in part on a faction's level of ground supremacy.[1] Ride hailing firms compete for both riders and drivers.[2] The marginal revenue from an additional driver depends in part on the firm's success marketing their platform to riders. Social media platforms compete for both users and advertisers. The marginal revenue from an additional user depends in part on the number of advertisers.[3] Pharmaceutical firms compete to convince both doctors and patients of their product's effectiveness.[4] The marginal revenue from persuading an additional patient may depend in part on the firm's success in convincing doctors.

This paper studies multi-battle contests where each agent simultaneously distributes a fixed budget between a finite number of battlefields. As in the Blotto contest of Borel (1921), contesting costs are sunk before agents allocate resources between battlefields. Each battlefield has a single prize which is divided among the competitors in proportion to a power function of the corresponding investment levels. Each agent seeks to maximize an objective function with constant subunitary elasticity of substitution between prize shares. A unique pure strategy Nash equilibrium is shown to exist under which resource allocations are proportional to prize values.

The unique Nash equilibrium is shown to be Pareto efficient over the set of feasible outcomes. Nonequilibrium outcomes often give every agent a lower payoff than they earn in equilibrium. No strategy profile gives every agent a higher payoff than they earn in equilibrium. Both the "size of the pie" and the "division of the pie" can vary, so these contests are not zero-sum games. In the two-agent case, equilibrium payoffs are shown to be minimax payoffs, so any deviation from equilibrium by one agent can be exploited by another

[1] Pirnie et al. (2005) discusses complementarity between air and ground supremacy.
[2] Farris et al. (2014) discusses competition for drivers between ride sharing firms.
[3] Fulgoni and Lipsman (2014) describes complementary users and advertisers.
[4] See Hurwitz and Caves (1988) for more on rent seeking by pharmaceutical firms.

to obtain an above-equilibrium payoff.

If battlefield success functions are sufficiently sensitive to investment levels, conventional Blotto contests with linear objectives have only mixed strategy equilibria (Roberson, 2006; Xu and Zhou, 2018). In contrast, the present paper shows that Blotto contests with arbitrarily sensitive battlefield success functions have pure strategy Nash equilibria if objective functions exhibit constant subunitary elasticity of substitution between prize shares.

The remainder of this paper is organized as follows. Section 2 discusses the related literature. Section 3 describes the contest. Section 4 considers the best response correspondence. Section 5 characterizes the Nash equilibrium. Section 6 discusses the efficiency of equilibrium and section 7 concludes. The proofs of all formal results in the text are provided in the appendix.

2 Related Literature

Much of the previous literature on multi-battle contests considers winner-take-all battles. In contrast, the present paper considers battles for shares of divisible prizes. Previous literature considers multi-battle contests with linear objectives, majoritarian objectives, weakest-link objectives, and best-shot objectives. In contrast, the present paper considers nonlinear objectives with constant elasticity of substitution between prize shares.

The remainder of this section highlights key differences between the present paper and the related literature on multi-battle contests. Borel (1921) describes the standard Blotto contest where each agent allocates limited resources over multiple battlefields. In this setting, battlefield success functions take the auction form under which the agent who allocates the most resources to a given battle wins it with certainty. Agents aim to maximize the number of battles they win. Roberson (2006) notes that such contests have no pure strategy Nash equilibria.

Xu and Zhou (2018) consider Blotto contests with battlefield success functions of the lottery form under which an agent's probability of success in a given battle depends on their investment levels. In contrast, the present paper considers battles for shares of divisible prizes. Xu and Zhou (2018) show that Blotto contests with lottery success functions and linear objectives have no pure strategy Nash equilibrium if battlefield success probabilities are sufficiently sensitive to investment levels. In contrast, the present paper shows that Blotto contests for shares of divisible prizes have pure strategy Nash equilibria under arbitrarily sensitive battlefield success if objective functions exhibit subunitary elasticity of substitution between prize shares.

Friedman (1958) considers Blotto contests with divisible prizes. He assumes that prizes shares are directly proportional to investment levels. In contrast, the present paper allows prize shares to be arbitrarily sensitive to investment levels. Friedman (1958) assumes that objective functions are linear in prize shares. In contrast, the present paper considers nonlinear objective functions with constant elasticity of substitution between prize shares.

Sela and Erez (2013) consider sequential Blotto contests with linear objectives and lottery battlefield success functions. Equilibrium resource allocations depend on both prize values and cost linkages across stages. If the prize value is the same in each stage, equilibrium resource allocations shown to be weakly decreasing over stages. Sela and Erez (2013) assume that objective functions are linear in battlefield success. In contrast, the present paper considers nonlinear objective functions with constant elasticity of substitution between prize shares. Sela and Erez (2013) assume that battlefield success is proportional to investment levels. In contrast, the present paper considers arbitrarily sensitive battlefield success functions.

Duffy and Matros (2015) consider Blotto contests with majoritarian objectives and lottery battlefield success functions. In settings with less than five battlefields, equilibrium investments are shown to be proportional to the number of winning subsets in which a given prize is pivotal. Anbarcı et al. (2020) consider sequential Blotto contests with majoritarian objectives and lottery battlefield success functions. In contests with at least four battlefields, prize valuations are shown to exist under which proportional resource allocation is not a Nash equilibrium. Klumpp et al. (2019) consider sequential Blotto contests with majoritarian objectives and lottery battlefield success functions. They identify pure strategy Nash equilibria under which players allocate their budget evenly across battlefields. Klumpp et al. (2019) assume that battlefield success functions exhibit limited sensitivity to investment levels. In contrast, the present paper considers arbitrarily sensitive battlefield success functions. Agents with majoritarian objectives aim to maximize the probability that they win a weighted majority of battles. In contrast, the present paper considers battles for shares of divisible prizes and objective functions with constant elasticity of substitution between prize shares.

Clark and Konrad (2007) consider two-player multi-battle contests with lottery success functions. The attacker has a best-shot objective and the defender has a weakest-link objective. The attacker wins the overall contest if they are successful in at least one battlefield. The defender only wins the overall contest if they are successful in every battlefield. Clark and Konrad (2007) assume that battlefield success probabilities are proportional to investment levels. In contrast, the present paper considers arbitrarily sensitive battlefield success functions. Clark and Konrad (2007) assume linear investment costs with no upper bound on total investments. In contrast, the present paper considers Blotto contests where each agent allocates limited resources over multiple battlefields.

Kovenock and Roberson (2018) consider multi-battle contests over networks of battlefields with success auction functions. In some networks, the defender has a weakest-link objective. In others, the defender has best-shot objectives. The defender wins if they successfully defend every network. The attacker wins if the defender fails to defend at least one network. Kovenock and Roberson (2018) consider linear investment costs with no upper bound on total investments. In contrast, the present paper considers Blotto contests where each agent allocates limited resources between multiple battlefields. Kovenock and Roberson (2018) assume that the agent who invests the most resources in a given battlefield wins it with certainty. In contrast, the present paper considers battles for shares of divisible prizes.

3 The Contest

Consider a multi-battle contest where $n \geq 2$ agents simultaneously distribute a fixed budget over m battlefields. The set of agents is indexed by $N = \{1, 2, \ldots, n\}$. The set of battlefields is indexed by $B = \{1, 2, \ldots, m\}$. Each battlefield contains a single divisible prize, so the set of prizes is also indexed by the set B. Let $v_b \in \mathbb{R}_{++}$ denote the value of the prize in battlefield b.

Let $x_{ib} \in \mathbb{R}_+$ denote the quantity of competitive resources invested by agent i in battlefield b. As in the Blotto contest of Borel (1921) and Roberson (2006), agent i's total investment w_i is sunk before they distribute their resources between battlefields. Agent i's allocation vector $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}^m_+$ satisfies the budget constraint

$$\sum_{b=1}^{m} x_{ib} = w_i \tag{1}$$

Let $X_i = \left\{x_i \in \mathbb{R}^m_+ : \sum_{k=1}^{m} x_{ib} = w_i\right\}$ denote agent i's strategy set and let $X = \prod_{i \in N} X_i$ denote the set of all possible strategy profiles. Let $y_{ib}(x)$ denote agent i's share of the prize in battlefield b under the strategy profile x. Let $\gamma_b : X \to \triangle_{n-1}$ denote the battlefield success function such that $\gamma_{bi}(x) = y_{ib}(x)$. The battlefield success function γ_{bi} is assumed to be increasing in x_{ib} and decreasing x_{jb} for $j \neq i$. It is assumed to be continuous, homogeneous of degree zero, and independent from irrelevant alternatives such that

$$\gamma_{bi}(x') = \frac{\gamma_{bi}(x)}{1 - \gamma_{bk}(x)} \quad \text{if } x'_{kb} = 0 \text{ and } x'_{jb} = x_{jb} \text{ for } j \in N \setminus \{i, k\}$$

Clark and Riis (1998) show that all such battlefield success functions must be proportional to a power function of the corresponding investment levels. Let $\mu_i \in \mathbb{R}_{++}$ denote the strength of agent i's competitive resources. Let $a \in \mathbb{R}_{++}$ denote the sensitivity of the battlefield success function to investment levels such that agent i's share of the prize in battlefield b is

$$y_{ib}(x) = \gamma_{bi}(x) = \frac{\mu_i x_{ib}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a} \qquad \text{if } \sum_{j=1}^{n} x_{jb} > 0 \tag{2}$$

If zero competitive resources are allocated to battlefield b, then $y_{ib}(x) = \gamma_{bi}(x) = \mu_i / \sum_{j=1}^{n} \mu_j$. If a is very small then prize shares are largely insensitive to resource allocations. Conversely, if a is very large then nearly the entirety of the prize in battlefield b is awarded to the agent who allocates the most resources to battlefield b. Agent i's battlefield success vector is $y_i(x) = (y_{i1}(x), \ldots, y_{im}(x)) \in \mathbb{R}_{+}^{m}$.

Each agent aims to maximize an objective function with constant elasticity of substitution between prize shares. If $y_i(x) \notin \mathbb{R}_{++}^{m}$ then agent i's payoff is $\pi_i(x) = 0$. As shown in the appendix, continuity implies that $\pi_i(x) = 0$ if $y_i(x) \notin \mathbb{R}_{++}^{m}$. Let $c_i \in \mathbb{R}_{+}$ denote the degree of complementary between prizes for agent i. If $y_i(x) \in \mathbb{R}_{++}^{m}$ then agent i's objective function is

$$\pi_i(x) = \left(\sum_{b=1}^{m} v_b y_{ib}(x)^{-c_i} \right)^{-\frac{1}{c_i}} \tag{3}$$

Uzawa (1962) shows that constant elasticity of substitution between prize shares implies the functional form of equation (3). For notational simplicity, dependence on the strategy profile x is sometimes suppressed. The sum of the prize values is normalized to unity. This is without loss of generality since linearly scaling the prize values by an arbitrary constant $\lambda \in \mathbb{R}_{++}$ linearly scales the objective function by a corresponding constant.

$$\left(\sum_{b=1}^{m} \lambda v_b y_{ib}^{-c_i} \right)^{-\frac{1}{c_i}} = \lambda^{-\frac{1}{c}} \left(\sum_{b=1}^{m} v_b y_{ib}^{-c_i} \right)^{-\frac{1}{c_i}} = \lambda^{-\frac{1}{c_i}} \pi_i \tag{4}$$

Agent i's elasticity of substitution between prizes is $\eta_i = (1 + c_i)^{-1}$ (Uzawa, 1962). Since the level of complementary $c_i \in \mathbb{R}_{+}$ is non-negative, the elasticity of substitution is less than one. Acemoglu (2002) and León-Ledesma et al. (2010) note that factors of production are complementary precisely when their elasticity of substitution is less than one. In the limit as $c_i \to \infty$, prizes are perfect complements and agent i's objective function converges to $\pi_i(x) = \min\{y_{i1}(x), \ldots, y_{im}(x)\}$. In this limiting case, there are multiple pure strategy Nash equilibria as illustrated by example 1.

Example 1. Consider a contest with two agents and two prizes where $n = m = 2$, $w_1 = w_2$, and $\mu_1 = \mu_2$. If $x_1 = x_2 = (\theta, w_i - \theta)$ with $\theta \in (0, w_i)$ then $y_{ib} = \frac{1}{2}$. In the limit as $c_i \to \infty$, prizes are perfect complements, so any unilateral deviation would be unprofitable as it would give the deviator less of at least one prize.

4 Best Responses

If any agent allocates a non-zero quantity of resources to battlefield b, then agent i's prize share y_{ib} is continuous in their allocation x_{ib} by equation (2). The objective function (3) is therefore continuous over the interior of the strategy set since it is continuous in prize shares. As illustrated by example 2, if all n agents allocate zero resources to battlefield b then agent i could obtain the entirety of prize b by reallocating an arbitrarily small quantity of resources to it.

Example 2. Consider a contest with two agents and two battlefields where $a = 1$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, $w = \mu = c = (1, 1)$. Suppose both players allocate all of their resources to battlefield 1, so $x_1 = x_2 = (1, 0)$. Then agent 1's battlefield success vector is $y_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the payoff to agent 1 is $\pi_1 = \frac{1}{2}$. If agent 1 reallocates a small portion ε of their resources from battlefield 1 to battlefield 2 then their battlefield success vector would equal $y_1' = \left(\frac{1-\varepsilon}{2-\varepsilon}, 1\right)$ and their payoff would equal $\pi_1' = \left(\frac{1}{2}\left(\frac{2-\varepsilon}{1-\varepsilon}\right) + \frac{1}{2}\right)^{-1}$. Taking the limit as ε converges to zero obtains $\lim\limits_{\varepsilon \to 0} \pi_1' = \frac{2}{3} > \frac{1}{2} = \pi_1$.

Proposition 1 states that agent i's objective function is strictly quasiconcave over the interior of the strategy set. Since the objective function is differentiable over this region, first order conditions are sufficient for maximization over the interior of the strategy set.

Proposition 1. *π_i is strictly quasiconcave over $x_i \in \mathbb{R}^n_{++}$.*

Proof. See Appendix A. $\square$

Strict quasiconcavity holds for $c_i > 0$ because agent i's payoff is then a strictly increasing function of the strictly concave function

$$g_i(x_i) = -\frac{1}{c_i}\sum_{b=1}^{m} v_b y_{ib}(x)^{-c_i} \tag{5}$$

Strict quasiconcavity also holds for the limiting case $c_i = 0$ because agent i's payoff is then a strictly increasing function of the strictly concave function

$$\log \pi_i = \sum_{b=1}^{m} v_b \log y_{ib} \tag{6}$$

Proposition 2 states that every resource allocation on the boundary of agent i's strategy set yields a strictly lower payoff than some other allocation in the interior of their strategy set. Hence the boundary of the strategy set never contains a best response. Since agent i's payoff is strictly quasiconcave over the interior of their strategy set, they cannot have multiple best responses. Taking a convex combination between any two distinct best responses would yield a larger payoff.

Proposition 2. *For every strategy profile $x \in X$ such that $x_{ib} = 0$ for some $b \in B$ there exists $x_i' \in X_i$ such that $\pi_i(x_i', x_{-i}) > \pi_i(x)$.*

Proof. See Appendix A. □

Proposition 2 implies that every best response lies in the interior of the strategy set. Agent i would obtain zero share of prize b if they allocate zero resources to the battlefield b and some other agent allocates a strictly positive amount. Conversely, if no other agent allocates resources to battlefield b, then agent i could obtain the entirety of prize b by allocating an arbitrarily small amount of resources to it. Proposition 3 characterizes the best response. Agent i maximizes their payoff by equalizing the marginal benefit of investment in each battlefield.

Proposition 3. *A strategy $x_i \in X_i$ maximizes agent i's payoff π_i if and only* *if for all battlefields $b, k \in B$*

$$\frac{v_b\left(1-y_{ib}\right)}{y_{ib}^{c_i}x_{ib}}=\frac{v_k\left(1-y_{ik}\right)}{y_{ik}^{c_i}x_{ik}} \tag{7}$$

Proof. See Appendix A. □

These conditions are both necessary and sufficient for payoff maximization because best responses are always unique and they always lie in the interior of the strategy set. If agent i's marginal payoff from investment in battlefield k was higher than their marginal payoff from investment in battlefield b then agent i could achieve a higher payoff by reallocating resources from battlefield b to battlefield k. Rearranging (7) to isolate the allocation ratio yields

$$\frac{x_{ib}}{x_{ik}}=\frac{v_b y_{ib}^{-c_i}\left(1-y_{ib}\right)}{v_k y_{ik}^{-c_i}\left(1-y_{ik}\right)} \tag{8}$$

Since c_i is non-negative, the right hand side of equation (8) is decreasing in y_{ib} and increasing in y_{ik}. If agent i is best responding and their share y_{ib} of prize b is larger than their share y_{ik} of prize k, then their allocation ratio x_{ib}/x_{ik} must be less than the corresponding value ratio v_b/v_k. Conversely, if y_{ib} was smaller than y_{ik} then x_{ib}/x_{ik} must be greater than v_b/v_k. Example 3 illustrates why a best response may fail to exist if an agent allocates all their resources to a single battlefield.

Example 3. Consider a contest with two agents and two battlefields where $a = 1$, $c_1 = c_2 = 1$, $\mu_1 = \mu_2$, and $v_1 = v_2$. Suppose agent 1 allocates all of their resources to battlefield 1 such that $x_1 = (0, 1)$. Then agent 2 can obtain the entirety of prize 2 by allocating an arbitrarily small quantity resources to battlefield 2. Hence $\pi_2\left(x_1, x_2\right) < \pi_2\left(x_1, x_2'\right)$ where $x_2 = (1-\varepsilon, \varepsilon)$ and $x_2' = \left(1-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)$ for all $\varepsilon \in (0, 1)$. Hence no interior strategy can be a best response for agent 2. Yet proposition 2 states that any best response must lie in the interior of the strategy space, so agent 2 has no best response to $x_1 = (0, 1)$.

If c_i was less than zero, the objective function could fail to be quasiconcave. In such settings, best responses might lie on the boundary of the strategy set and agents might have multiple best responses. Example 4 considers a setting where $c_i = -2$ and one agent has multiple best responses on the boundary of their strategy set.

Example 4. Consider a contest with two agents and two battlefields where $c_1 = c_2 = -2$, $n = m = 2$, $a = 2$, $\mu_1 = \mu_2 = 1$, and $v_1 = v_2$. The resulting payoff function is $\pi_i(x) = v_1^{1/2}\left(y_{i1}(x)^2 + y_{i2}(x)^2\right)^{1/2}$. In this case, receiving the entirety of one prize and none of the other is better than receiving half of each prize. If $x_1 = (0.5, 0.5)$, then agent 2 can maximize their objective function by either allocating all of their resources to battlefield 1 or by allocating all of their resources to battlefield 2.

5 Nash Equilibrium

The marginal value of an increase in agent i's share of one prize depends on their share of other prizes. Complementarity between prizes incentivizes agent i to invest more in battlefields where they are relatively less successful. Proposition 4 states that agent i must receive the same share of each prize in equilibrium.

Proposition 4. *In every pure strategy Nash equilibrium, $y_{ib} = y_{ik}$ for every agent i and all battlefields b and k.*

Proof. See Appendix A. □

If agent i's share of prize b was larger than their share of prize k then some other agent j must have a larger share of prize k than prize b. Then agent i's allocation ratio x_{ib}/x_{ik} would be greater than agent j's allocation ratio x_{jb}/x_{jk} by equation (2). If both agents are best responding, equation (8) would then imply that agent i's allocation ratio must be lower than agent j's allocation ratio. Hence agent i must obtain the same share of each prize in equilibrium and equation (8) implies that the equilibrium allocation ratio must equal the corresponding value ratio. Theorem 1 characterizes the Nash equilibrium.

Theorem 1. *The unique pure strategy Nash equilibrium is* $x^*_{ib} = w_i v_b$.

Proof. See Appendix A. □

The Nash equilibrium strategy profile depends on neither the level of complementarity c_i nor on the sensitivity a of the battlefield success function. Yet proposition 3 implies that agent i's best response generally depends on both of these parameters. In equilibrium, $y_{ib} = y_{ik}$ by proposition 4 and equation (8) reduces to

$$\frac{x_{ib}}{x_{ik}} = \frac{v_b}{v_k} \tag{9}$$

In this case, all of the terms involving a and c_i cancel out, so neither parameter affects the equilibrium investment ratio. The elasticity of substitution between prizes is given by $\eta_i = (1 + c_i)^{-1}$ (Uzawa, 1962). Hence c_i is positive precisely when the elasticity of substitution η_i is less than one. As discussed in as stated in section 3, theorem 1 always holds when $c_i \geq 0$. Example 5 shows that the proportional strategy profile described by theorem 1 is still a Nash equilibrium in some contests with super-unitary elasticity of substitution.

Example 5. If $w_1 = w_2 = \mu_1 = \mu_2 = 1$, $v_1 = v_2 = \frac{1}{2}$, and $\eta_1 = \eta_2 = 2$ then $c_1 = c_2 = -\frac{1}{2}$ and

$$\pi_i(x) = \left(\frac{1}{2} y_{i1}(x)^{\frac{1}{2}} + \frac{1}{2} y_{i2}(x)^{\frac{1}{2}}\right)^2 = g_i(x)^2 \tag{10}$$

$$\text{where } g_i(x) = \frac{1}{2} y_{i1}(x)^{\frac{1}{2}} + \frac{1}{2} y_{i2}(x)^{\frac{1}{2}} \tag{11}$$

If $a = 1$ then $y_{ib}(x)$ is strictly concave in x_{ib} so $g_i(x)$ is strictly concave in x_i and π_i is strictly quasiconcave in x_i. Hence the first order condition is sufficient for maximization and the proportional strategy profile $x_{ib} = \frac{1}{2}$ is a Nash equilibrium.

Example 6 illustrates why subunitary elasticity of substitution between prizes is important for theorem 1. For every super-unitary elasticity of substitution

$\eta_i > 1$ there are contests where the proportional strategy profile is not a Nash equilibrium.

Example 6. If $w_1 = w_2 = \mu_1 = \mu_2 = 1$ and $c_1 < 0$ then

$$\pi_1 (x) = \left(v_1 y_{i1} (x)^{-c_1} + v_2 y_{i2} (x)^{-c_1}\right)^{-\frac{1}{c_1}} \tag{12}$$

If $v_1 = \left(\frac{2}{3}\right)^{-c_1}$, $v_2 = 1 - v_1$, and $x_1 = x_2 = (v_1, v_2)$ then $\pi_1 (x_1, x_2) = \frac{1}{2}$. If $x_1' = (1, 0)$ then

$$\lim_{a \to \infty} \pi_1 (x_1', x_2) = v_1^{-\frac{1}{c_1}} = \frac{2}{3} > \frac{1}{2} = \lim_{a \to \infty} \pi_1 (x_1, x_2) \tag{13}$$

Hence x_1 is not a best response to x_2 if battlefield success functions are sufficiently sensitive.

6 Efficiency

Proposition 5 states that agent i's equilibrium payoff is a function of their endowment w_i. Equilibrium payoffs exhibit greater sensitivity to initial endowments when battlefield success functions exhibit greater sensitivity to investment levels. Less sensitive battlefield success functions make equilibrium payoffs less sensitive to initial endowments without distorting equilibrium allocations.

Proposition 5. *The unique Nash equilibrium payoff to agent i is*

$$\pi_i^* = \frac{\mu_i w_i^a}{\sum_{\ell=1}^n \mu_\ell w_\ell^a} \tag{14}$$

Proof. See Appendix A. □

Proposition 6 states that Nash equilibrium maximizes the total payoff to all n agents. Consequently, the equilibrium strategy profile is Pareto efficient over the set of feasible outcomes. Hence any non-equilibrium strategy profile that gives one agent a greater payoff than they earn in equilibrium must give some other agent a lower payoff than they earn in equilibrium.

Proposition 6. *The maximum total payoff to all n agents over all feasible strategy profiles $x \in X$ is*

$$\max_{x \in X} \sum_{i=1}^{n} \pi_i (x) = 1$$

Proof. See Appendix A. □

The Pareto efficiency of the unique Nash equilibrium immediately rules out the possibility of strategy profiles that give every agent a higher payoff than they earn in equilibrium. Proposition 7 states that, in the two agent case, either agent can obtain an above-equilibrium payoff if their opponent employs a non-equilibrium strategy. Consequently, Nash equilibrium payoffs are minimax payoffs in the two agent case.

Proposition 7. *If agent j employs a non-equilibrium strategy and $n = 2$ then agent i can obtain an above-equilibrium payoff.*

Proof. See Appendix A. □

Many non-equilibrium outcomes are Pareto dominated by the equilibrium outcome. Example 7 illustrates how non-equilibrium strategy profiles can give every player a lower payoff than they earn in equilibrium. Both the "size of the pie" and the "division of the pie" can vary, so these contests are not zero-sum games.

Example 7. Consider a contest with two players and two battlefields where $a = 1$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $w = \mu = c = (1, 1)$. If $x_1 = x_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ then $y_1 (x) = y_2 (x) = \left(\frac{1}{2}, \frac{1}{2}\right)$. If $x'_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and $x'_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$ then $y_1 (x') = \left(\frac{1}{3}, \frac{2}{3}\right)$ and $y_2 (x') = \left(\frac{2}{3}, \frac{1}{3}\right)$ so

$$\pi_i (x') = \left(\frac{1}{2y_{i1} (x')} + \frac{1}{2y_{i2} (x')} \right)^{-1} \tag{15}$$

$$\pi_1 (x') = \pi_2 (x') = \frac{4}{9} < \frac{1}{2} = \pi_1 (x) = \pi_2 (x) \tag{16}$$

7 Conclusion

This paper considers multi-battle conflicts where agents allocate fixed budgets to compete over divisible prizes. The share of a given prize awarded to a given agent is given by an arbitrarily sensitive battlefield success function. Prize shares are proportional to a power function of investments and payoffs exhibit constant elasticity in prize shares. Blotto contests with linear objective functions have no pure strategy Nash equilibrium if success functions are sufficiently sensitive to investment levels. Conversely, if prizes exhibit subunitary elasticity of substitution, the proportional strategy profile is shown to be the unique Nash equilibrium under arbitrarily sensitive success functions.

The unique Nash equilibrium is shown to be Pareto efficient over the set of feasible outcomes. No strategy profile gives every player a higher payoff than they earn in equilibrium and nonequilibrium strategy profiles often give every player a lower payoff than they earn in equilibrium. Both the "size of the pie" and the "division of the pie" can vary, so these contests are not zero-sum games. In the two agent case, equilibrium payoffs are shown to be minimax payoffs. Any deviation from equilibrium by one agent can be exploited by the other to obtain an above-equilibrium payoff.

Many settings involve competition over complementary prizes. Ride hailing firms compete for both riders and drivers. Pharmaceutical firms compete to persuade both doctors and patients. Military factions compete for both air supremacy and ground supremacy. The existence of pure strategy Nash equilibria in such settings may depend on the level of complementarity between prizes. If prizes are sufficiently complementary, pure strategy Nash equilibria can exist under arbitrarily sensitive success functions. In contrast, Blotto contests where payoffs are linear in battlefield success have only mixed strategy Nash equilibria if success functions are sufficiently sensitive to investment levels. If policy makers are risk averse, they may prefer pure strategy equilibria over mixed strategy equilibria. Complementarity between prizes can allow pure strategy Nash equilibria to persist in the presence of highly sensitive success functions.

Generalizations of the present model should be considered by future research.

The present paper considers objective functions that exhibit constant subunitary elasticity of substitution between prize shares. Example 6 shows that there are prize valuations under which the proportional strategy profile is not a Nash equilibrium if the elasticity of substitution between prize shares is greater than one. Future research should provide a more complete characterization of Nash equilibria in settings where the elasticity of substitution between prizes is greater than one. The present paper allows different prizes to have different values but assumes that a given prize has the same value to every agent. Future research should consider settings where different agents have different values for the same prize. The present paper assumes that prize shares are equally sensitive to investment levels in each battlefield. Future research should consider settings where different battlefield success functions exhibit different levels of sensitivity.

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A Proofs

Proof of Payoff Continuity. If $y_i \in \mathbb{R}^m_{++}$then the payoff to agent i is

$$\pi_i = \left(\sum_{b=1}^{m} v_b y_{ib}^{-c_i}\right)^{-\frac{1}{c_i}} = \frac{1}{\left(\sum_{b=1}^{m} \frac{v_b}{y_{ib}^{c_i}}\right)^{c_i}} \tag{17}$$

Since $c_i > 0$ the fraction $v_b / y_{ib}^{c_i}$ increases without bound as y_{ib} converges to zero, so the sum $\left(\sum_{b=1}^{m} \frac{v_b}{y_{ib}^{c_i}}\right)^{c_i}$ increases without bound as as y_{ib} converges to zero. Hence π_i converges to zero as y_{ib} converges to zero. $\square$

Proof of Proposition 1. Let g_i denote an increasing function of π_i given by

$$g_i = -\frac{1}{c\pi_i^{c_i}} = -\frac{1}{c_i} \sum_{b=1}^{m} v_b y_{ib}^{-c_i} \tag{18}$$

Differentiating y_{ib} with respect to x_{ib} yields

$$\begin{aligned} \frac{\partial y_{ib}}{\partial x_{ib}} &= \frac{\partial}{\partial x_{ib}} \left[\frac{\mu_i x_{ib}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right] = \frac{\sum_{j \neq i} \mu_j x_{jb}^a}{\left(\sum_{j=1}^{n} \mu_j x_{jb}^a\right)^2} a \mu_i x_{ib}^{a-1} \\ &= \frac{a}{x_{ib}} \left(\frac{\sum_{j \neq i} \mu_j x_{jb}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right) \left(\frac{\mu_i x_{ib}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right) = \frac{a\left(1 - y_{ib}\right) y_{ib}}{x_{ib}} \end{aligned}$$

So differentiating g_i with respect to x_i yields

$$\frac{\partial g_i}{\partial x_{ib}} = \frac{\partial g_i}{\partial y_{ib}} \frac{\partial y_{ib}}{\partial x_{ib}} = \frac{v_b}{y_{ib}^{1+c_i}} \frac{a\left(1 - y_{ib}\right) y_{ib}}{x_{ib}} = \frac{a v_b \left(1 - y_{ib}\right)}{y_{ib}^{c_i} x_{ib}} \tag{19}$$

Since the numerator of (19) is decreasing in x_{ib} and the denominator is increasing in x_{ib} we have $\frac{\partial^2 g_i}{\partial x_{ib}^2} < 0$. Since (19) is constant in x_{ih} for $h \neq b$, the mixed second order partial derivatives are given by $\frac{\partial^2 g_i}{\partial x_{ib} \partial x_{ih}} = 0$. Thus the matrix of second order partial derivatives is negative definite, so g_i is strictly concave in x_i. Hence π_i is strictly quasiconcave in x_i for $c_i > 0$ since g_i is a strictly increasing function of π_i. If $c_i = 0$ then the payoff to agent i is given by

$$\pi_i = \prod_{b=1}^{m} y_{ib}^{v_b} \tag{20}$$

Taking the logarithm of both sides obtains

$$\log \pi_i = \sum_{b=1}^{m} v_b \log y_{ib} \tag{21}$$

Differentiating with respect to x_{ib} yields

$$\frac{\partial}{\partial x_{ib}}\left[\log \pi_i\right] \quad = \quad \frac{v_b}{y_{ib}}\frac{\partial y_{ib}}{\partial x_{ib}} = \frac{a\left(1-y_{ib}\right)}{x_{ib}} \tag{22}$$

Thus $\frac{\partial^2 \log \pi_i}{\partial x_{ib}^2} < 0$ and $\frac{\partial^2 \log \pi_i}{\partial x_{ib} \partial x_{ih}} = 0$ for $b \neq h$. Hence π_i is strictly quasiconcave in x_i for $c_i = 0$. □

Proof of Proposition 2. Let $x \in X$ such that $x_{ib} = 0$. Now consider the alternative strategy $\hat{x}_i \in X_i$ such that

$$\hat{x}_{ik} = \varepsilon \frac{w_i}{m} + (1-\varepsilon)\, x_{ik} \tag{23}$$

If $x_{jb} > 0$ for some $j \neq i$ then $\pi_i(x) = 0 < \pi_i(\hat{x}_i, x_{-i})$. Alternatively, if $x_{jb} = 0$ for all j then $\gamma_{bi}(x) = \mu_i / \sum_{j=1}^{n} \mu_j < 1 = y_{ib}(\hat{x}_i, x_{-i})$ for all $\varepsilon > 0$. Since $x_{jb} = 0$ for all $j \in N$ there exists at least one battlefield $h \in B$ such that $x_{jh} > 0$ for some $j \neq i$. Hence the limiting value of $\gamma_{hi}(\hat{x}_i, x_{-i})$ as ε approaches zero from above is given by

$$\lim_{\varepsilon \downarrow 0} \gamma_{hi}(\hat{x}_i, x_{-i}) = \frac{\mu_i x_{ih}}{\sum_{j=1}^{n} \mu_j x_{jh}^{a}} = y_{ih}(x) \tag{24}$$

Hence $\pi_i(\hat{x}_i, x_{-i}) > \pi_i(x)$ for some $\varepsilon > 0$ since π_i is continuous over $y_i \in \mathbb{R}_{++}^{m}$. □

Proof of Proposition 3. Suppose x_i is a best response for agent i. By proposition 2, x_i must lie in the interior of the strategy set. Since agent i's payoff is differentiable over the interior of the strategy set we have

$$\frac{\partial \pi_i}{\partial x_{ib}} \quad = \quad \frac{\partial \pi_i}{\partial x_{ik}} \tag{25}$$

$$\frac{v_b\left(1-y_{ib}\right)}{y_{ib}^{c_i} x_{ib}} \quad = \quad \frac{v_k\left(1-y_{ik}\right)}{y_{ik}^{c_i} x_{ik}} \tag{26}$$

By proposition 2 every best response must lie in the interior of the strategy set. By proposition 1 agent i's payoff π_i is strictly quasiconcave over this region. Since the objective function is differentiable in x_i over this region, the first order conditions on x_i are sufficient for maximization. $\square$

Proof of Proposition 4. If $y_{ib} < y_{ik}$ then we have

$$\sum_{\ell \neq i} y_{\ell b} = (1 - y_{ib}) > (1 - y_{ik}) = \sum_{\ell \neq i} y_{\ell k} \tag{27}$$

Hence there exists $j \neq i$ such that $y_{jk} < y_{jb}$ and

$$y_{ib} y_{jk} < y_{ik} y_{jb} \tag{28}$$

$$\left(\frac{\mu_i x_{ib}^a}{Z_b}\right)\left(\frac{\mu_j x_{jk}^a}{Z_k}\right) < \left(\frac{\mu_i x_{ik}^a}{Z_k}\right)\left(\frac{\mu_j x_{jb}^a}{Z_b}\right) \quad \text{where } Z_b = \sum_{\ell=1}^{n} \mu_\ell x_{\ell b}^a \tag{29}$$

$$x_{ib} x_{jk} < x_{ik} x_{jb} \tag{30}$$

By proposition 3 the first order conditions on x_i can be written as

$$\frac{x_{ib}}{x_{ik}} = \frac{v_b (1 - y_{ib})\, y_{ib}^{-c_i}}{v_k (1 - y_{ik})\, y_{ik}^{-c_i}} \tag{31}$$

If x is a Nash equilibrium then by equation (31) we have

$$\begin{matrix} y_{ib} < y_{ik} \\ y_{jk} < y_{jb} \end{matrix} \implies \frac{x_{ib}}{x_{ik}} > \frac{v_b}{v_k} > \frac{x_{jb}}{x_{jk}} \implies x_{ib} x_{jk} > x_{ik} x_{jb} \tag{32}$$

But this contradicts equation (30). $\square$

Proof of Theorem 1. By proposition 4 if x is a Nash equilibrium strategy profile then for every agent i there exists $\bar{y}_i \in [0, 1]$ such that for every battlefield b agent i's share of prize b is $y_{ib} = \bar{y}_i$. Hence by proposition 3 the necessary and sufficient first order conditions on x_i for the maximization of π_i are given by

$$\frac{v_b (1 - \bar{y}_i)}{x_{ib} \bar{y}_i^{c_i}} = \frac{v_k (1 - \bar{y}_i)}{x_{ik} \bar{y}_i^{c_i}} \tag{33}$$

$$\frac{v_b}{v_k} = \frac{x_{ib}}{x_{ik}} \tag{34}$$

Hence $x_{ib} v_k = x_{ik} v_b$ and summing over k obtains $x_{ib} = w_i v_b$. $\square$

Proof of Proposition 5. By proposition 4 if x is a Nash equilibrium strategy profile then for every agent i there exists $\bar{y}_i \in [0, 1]$ such that for every battlefield b agent i's share of prize b is given by $y_{ib} = \bar{y}_i$. Hence the payoff to agent i can be written as

$$\pi_i = \left(\sum_{b=1}^{m} v_b \bar{y}_i^{-c_i} \right)^{-\frac{1}{c_i}} = \bar{y}_i \left(\sum_{b=1}^{m} v_b \right)^{-\frac{1}{c_i}}$$

Now since $\sum_{b=1}^{m} v_b = 1$ we have

$$\pi_i = \bar{y}_i = \frac{\mu_i w_i^a}{\sum_{\ell=1}^{n} \mu_\ell w_\ell^a} \tag{35}$$

□

Proof of Proposition 6. Let Y denote the set of all $y \in \mathbb{R}_+^{n\times m}$ such that for all battlefields $b \in B$ the sum of all prize shares is given by $\sum_{i=1}^{n} y_{ib} = 1$. Hence Y includes all feasible outcomes. Let g_i denote a strictly increasing function of π_i given by

$$g_i = -\frac{1}{c_i \pi_i^{c_i}} = -\frac{1}{c_i} \sum_{b=1}^{m} v_b y_{ib}^{-c_i} \tag{36}$$

For $\theta \in \triangle^{n-1}$ let G_θ denote a weighted sum of all g_i given by

$$G_\theta = \sum_{i=1}^{n} \theta_i g_i = -\sum_{i=1}^{n} \frac{1}{c_i} \sum_{b=1}^{m} \theta_i v_b y_{ib}^{-c_i} \tag{37}$$

Hence G_θ is increasing in π_i for each agent i. Now differentiating G_θ with respect to y_{ib} yields $\frac{\partial G_\theta}{\partial y_{ib}} = \frac{\theta_i v_b}{y_{ib}^{c_i+1}} > 0$ and twice differentiating G_θ with respect to y_{ib} yields $\frac{\partial^2 G_\theta}{\partial y_{ib}^2} = -(c_i+1)\frac{\theta_i v_b}{y_{ib}^{c_i+2}} < 0$. The cross partial derivatives are given by $\frac{\partial G}{\partial y_{ib} \partial y_{jb}} = 0$. Hence G_θ is strictly concave over $y_{Nb} = (y_{1b}, \ldots, y_{nb}) \in \mathbb{R}_{++}^m$. The first order conditions on y_{Nb} for the maximization of G_θ are given by

$$\frac{\theta_i v_b}{y_{ib}^{c_i+1}} = \frac{\partial G}{\partial y_{ib}} \quad = \quad \frac{\partial G}{\partial y_{jb}} = \frac{\theta_j v_b}{y_{jb}^{c_j+1}} \tag{38}$$

$$\frac{\theta_i}{\theta_j} = \frac{y_{ib}^{c_i+1}}{y_{jb}^{c_j+1}} \quad \Longrightarrow \quad y_{ib} = \bar{y}_i \tag{39}$$

Thus if y maximizes G_θ over Y then the payoff to agent i satisfies $\pi_i = \bar{y}_i$. Now if $y \in Y$ maximizes the total payoff $\sum_{i=1}^{n} \pi_{ib}$ over Y then it is Pareto efficient over Y, so there exists some $\theta \in \triangle^{n-1}$ such that y maximizes G_θ

over Y and the total payoff is given by

$$\sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} \bar{y}_i = 1$$

□

Proof of Proposition 7. Let x_j denote an allocation employed by agent j and suppose that agent i employs the allocation

$$x_{ib} = \frac{w_i x_{jb}}{w_j} \tag{40}$$

Then the share of prize b awarded to agent i is given by

$$y_{ib} = \frac{\mu_i x_{ib}^a}{\mu_i x_{ib}^a + \mu_j x_{jb}^a} = \frac{\mu_i w_i^a}{\mu_i w_i^a + \mu_j w_j^a} = \bar{y}_i \tag{41}$$

Hence the payoff to agent i is given by

$$\pi_i = \left(\sum_{b=1}^{m} v_b \bar{y}_i^{-c_i} \right)^{-\frac{1}{c_i}} = \bar{y}_i \left(\sum_{b=1}^{m} v_b \right)^{-\frac{1}{c_i}} = \frac{\mu_i w_i^a}{\mu_i w_i^a + \mu_j w_j^a} \tag{42}$$

Thus by proposition 5 agent i can always obtain at least their unique Nash equilibrium payoff. Now if $x_{jb} \neq w_j v_b$ then the strategy given by Equation (40) does not satisfy the first order conditions for the maximization of agent i's payoff. Hence it is not a best response by proposition 3, so there exists some alternative strategy that does better. □