Multi-battle Contests over Complementary Battlefields

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Abstract

Achieving success in one conflict can often enhance the value of success in other conflicts. This paper investigates multi-battle contests where asymmetrically endowed agents allocate resources to compete over multiple complementary prizes. The share of each prize awarded to each agent is given by an arbitrarily decisive contest success function. Prizes serve as constant elasticity inputs with with an arbitrary degree of complementarity. This value structure covers a wide variety of cases ranging from Cobb-Douglas to perfect complements. Such contests are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive success functions. In contrast, conventional blotto games and multi-battle conflicts have no pure strategy Nash equilibrium when contest success functions are sufficiently decisive. These results indicate that complementarity between prizes can play an important role in stabilizing strategic behavior.

1 Introduction

Success in one conflict often enhances the value of success in other conflicts. Firms that offer ride hailing services compete to market their platform to riders and recruit drivers.[1] The marginal revenue earned from recruiting an additional driver depends in part on the firm's success marketing their platform to riders.[2] Social media platforms compete for both users and advertisers. The marginal revenue from an additional user depends in part on the firm's success in obtaining advertisers.[3] Pharmaceutical firms compete to convince doctors of their product's effectiveness and to persuade patients to request it.[4] The marginal revenue from persuading an additional patient depends in part on the firm's success in convincing doctors.

This paper considers multi-battle contests where an arbitrary number of asymmetrically endowed agents allocate resources to compete over an arbitrary number of complementary battlefields. Each agent is endowed with a unidimensional stock of competitive resources which they allocate between battlefields. In each battlefield, agents compete over a divisible prize with a distinct value. The share of each prize awarded to each agent is given by an arbitrarily decisive success function. Prizes serve as constant elasticity inputs with an arbitrary degree of complementarity.

This prize value structure covers a wide variety of cases of ranging from Cobb-Douglass to perfect complements. The generality of the value structure is important because, in many applications, an agent's marginal value for a small increase in one prize varies gradually with her share of other prizes. Greater air superiority can help a military faction more effectively control a contested region, but the marginal control gained from a small increase in air superiority varies gradually with the faction's ground superiority.[5] Hiring additional drivers can help

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[1] Farris et al. (2014) provides details regarding the heavy competition for drivers between ride sharing firms. [2] See Jia and Mayer (2016) for more on complementarity between market advertising and political lobbying. [3] See Fulgoni and Lipsman (2014) for more on complementarities between user-base and advertisers on social media platforms. [4] See Hurwitz and Caves (1988) for more on rent seeking by pharmaceutical firms. [5] See Pirnie et al. (2005) for details regarding complementarity between air supremacy and ground supremacy in military conflicts.

a ride hailing firm earn more revenue, but the marginal revenue earned from an additional driver varies gradually with the number of riders.

These contests are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive contest success functions. Further, the infinitely repeated two agent contest is shown to possess a unique subgame perfect Nash equilibrium. In contrast, conventional blotto games and multi-battle conflicts have no pure strategy Nash equilibrium when contest success functions are sufficiently decisive. These results indicate that complementarity between prizes can play an important role in stabilizing strategic behavior.

The remainder of this paper is organized as follows. Section 2 discusses the related literature. Section 3 formally describes the multi-battle contest under investigation. Section 4 establishes key properties of agent i's best response correspondence. Section 5 demonstrates the existence of a unique pure strategy Nash equilibrium. Section 6 considers equity and efficiency of the Nash equilibrium outcome. Section 7 considers two stage multi-battle contests where agents make costly investments in the first stage and then allocate resources over battlefields in the second stage. Section 8 considers repeated multi-battle comtests. Section 9 concludes and discusses important implications of the results.

2 Related Literature

A significant potion of the previous literature on multi-battle contests considers prizes that are perfect substitutes. Friedman (1958) considers multi-battle contests where two firms make advertising expenditures to compete over sales in several distinct marketing areas. Robson (2005) investigates two-player multi-item contests between resource constrained agents where prizes are perfect substitutes and contest success functions are probabilistic. Roberson (2006) examines two-player blotto games with deterministic winner-take-all contest success functions. A survey of the multi-battle contest literature is provided by Kovenock and Roberson (2010).

A number of previous works consider specific instances of complementarity in multi-battle conflicts. Englmaier et al. (2009) identify asymmetric equilibria in two-bidder auctions over three items where a single item is has no value by itself and three items are worth no more than two items. Szentes and Rosenthal (2003) identify symmetric mixed strategy equilibria in auctions over three items where the marginal value increases for the second item and decreases for the third item. Rai and Sarin (2009) consider contests where agents make complementary investments to compete over a single prize. Kolmar and Rommeswinkel (2013) examine contests between teams of agents who exert complementary effort and face linear costs. Kovenock and Roberson (2017) consider two-player weakest-link conflicts with deterministic winner-take-all success functions where prizes are perfect substitutes for one agent and perfect complements for the opposing agent.

Clark and Konrad (2007) consider weakest-link contests with probabilistic Tullock success functions. Kovenock et al. (2015) consider two player Hex games where the winner of each cell is determined by a linear cost Tullock contest. Duffy and Matros (2017) examine two-player probabilistic blotto games where players seek to obtain a majority share of the overall prize value. Deck et al. (2017) investigate two-player multi-battle conflicts with linear effort costs where players seek to obtain a majority share of the overall prize value. In contrast, this paper considers a prize value structure where prizes serve as constant elasticity inputs with an arbitrary degree of complementarity. This prize value structure is covers a wide variety of cases of ranging from Cobb-Douglass to perfect complements.

Several previous works find that conventional blotto games and multi-battle conflicts have no pure strategy Nash equilibrium when success functions are sufficiently decisive. Baye et al. (1994) show that Tullock contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium if the contest success function is sufficiently decisive. Arbatskaya and Mialon (2010) show that two-player multi-activity contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium if the contest success function is sufficiently decisive.

Ewerhart (2017) shows that Nash equilibrium payoffs in probabilistic contests where agents expend effort at constant marginal cost converge to those of the mixed strategy Nash equilibrium in the deterministic winner-takes-all contest as the success function becomes increasingly decisive. Roberson (2006) notes that conventional blotto games have no pure strategy Nash equilibrium unless one player is strong enough to guarantee complete victory in every battlefield. In contrast, the multi-battle contests considered by this paper are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive contest success functions.

3 Complementary Battlefields

Consider a multi-battle conflict where n agents simultaneously allocate limited resources over m complementary battlefields. Let $N = \{1, \ldots, n\}$ denote the set of agents and $B = \{1, \ldots, m\}$ denote the set of battlefields. Agent $i \in N$ is endowed with a unidimensional stock $w_i \in \mathbb{R}_{++}$ of competitive resources. Let $x_{ib} \in \mathbb{R}_+$ denote the quantity of competitive resources that agent i devotes to battlefield b. The strategy $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}_+^m$ employed by agent i must satisfy the budget constraint

$$\sum_{k=1}^{m} x_{ib} = w_i \tag{1}$$

Let $X_i = \left\{x_i \in \mathbb{R}_+^m : \sum_{k=1}^{m} x_{ib} = w_i\right\}$ denote the set of agent i's feasible strategies and let $X = \prod_{i \in N} X_i$ denote the set of all feasible strategy profiles. In each battlefield b agents compete over a distinct divisible prize. If every agent allocates zero competitive resources to battlefield b, then agent i's share y_{ib} of prize b is given by $y_{ib}(x) = \frac{1}{n}$. Otherwise the share of prize b awarded to agent i is given by

$$y_{ib}(x) = \frac{x_{ib}^a}{\sum_{j=1}^{n} x_{jb}^a} \tag{2}$$

The parameter $a \in \mathbb{R}_{++}$ denotes the decisiveness of the battlefield success function. In the limit as $a \to \infty$ the entirety of prize b is awarded to the agent who allocates the most resources to battlefield b. Conversely, in the limit as $a \to 0$ prize b is divided equally among all the agents competing over battlefield b. Agent i's battlefield success vector is given by $y_i = (y_{i1}, \ldots, y_{im}) \in \mathbb{R}_+^m$. Each of the m prizes serves as a complementary input to agent i's payoff, which exhibits constant elasticity of substitution between prizes. If $y_i \notin \mathbb{R}^m_{++}$ then agent i's payoff is given by $\pi_i(y_i) = 0$.[6] Otherwise agent i's payoff π_i is given by[7]

$$\pi_i(y_i) = \beta \left(\sum_{b=1}^{m} v_b y_{ib}^{-c} \right)^{-\frac{1}{c}} \tag{3}$$

The payoff scale is given by $\beta \in \mathbb{R}_{++}$. The degree of complementary between battlefields i s given by $c \in \mathbb{R}_{++}$. In the limit as $c \to \infty$, all m prizes are perfect complements and agent i's payoff is given by $\pi_i(y_i) = \beta \min\{y_{i1}, \ldots, y_{im}\}$. Conversely, in the limit as $c \to 0$, the payoff to agent i takes the Cobb-Douglas form[8] $\pi_i(y_i) = \prod_{b=1}^{m} y_{ib}^{v_b}$. The share parameter $v_b \in \mathbb{R}_{++}$ denotes the relative value of prize b. The sum of all m share parameters is given by $\sum_{b=1}^{m} v_b = 1$ without loss of generality since

$$\beta \left(\sum_{b=1}^{m} \lambda v_b y_{ib}^{-c} \right)^{-\frac{1}{c}} = \lambda^{-\frac{1}{c}} \beta \left(\sum_{b=1}^{m} v_b y_{ib}^{-c} \right)^{-\frac{1}{c}} = \lambda^{-\frac{1}{c}} \pi_i \tag{4}$$

3.1 Continuity

Theorem 1 states that agent i's payoff π_i approaches zero as her share y_{ib} of prize b approaches zero. By definition, agent i's payoff equals zero whenever $y_i = (y_{i1}, \ldots, y_{im}) \notin \mathbb{R}^m_{++}$ so Theorem 1 implies that her payoff π_i is continuous in her success vector y_i.

Theorem 1. *The limiting value of agent i's payoff π_i as her allocation y_{ib} to battlefield b approaches zero is given by* $\lim_{y_{ib} \to 0} \pi_i(y_i) = 0$

[6] As shown by Theorem 1 below, continuity requires that $\pi_i(y_i) = 0$ for all $y_i \notin \mathbb{R}^m_{++}$. [7] See Uzawa (1962) for details regarding the necessity and sufficiency of this functional form for constant elasticity of substitution. [8] See Saito (2012) for a proof of convergence to the Cobb–Douglas form as c approaches zero.

Proof. If $y_i \in \mathbb{R}^m_{++}$then the payoff to agent i is given by

$$\pi_i\left(y_i\right) = \beta\left(\sum_{b=1}^{m} v_b y_{ib}^{-c}\right)^{-\frac{1}{c}} = \frac{\beta}{\left(\sum_{b=1}^{m}\frac{v_b}{y_{ib}^c}\right)^c} \tag{5}$$

Since $c > 0$ the fraction v_b/y_{ib}^c increases without bound as y_{ib} converges to zero, so the sum $\left(\sum_{b=1}^{m}\frac{v_b}{y_{ib}^c}\right)^c$ increases without bound as as y_{ib} converges to zero. Hence $\pi_{ib}\left(y_i\right)$ converges to zero as y_{ib} converges to zero. □

If any agent allocates a non-zero quantity of resources to battlefield b then agent i's share y_{ib} of prize b is continuous in her allocation x_{ib} as given by Equation (2), so Corollary 1 states that agent i's payoff is continuous in her strategy over the interior of her strategy set.

Corollary 1. *Agent i's payoff π_i is continuous in her strategy x_i over the interior of her strategy set* $\text{int}(X_i) = X_i \cap \mathbb{R}^m_{++}$.

Continuity of an agent's payoff in her success vector does not necessarily imply continuity in her strategy over her entire strategy set. As illustrated by Example 1 an agent's payoff can be discontinuous in her strategy profile on the boundary of her strategy set. In particular, if all agents allocate zero resources to battlefield b then agent i can obtain the entirety of prize b by reallocating an arbitrarily small portion of her resources to battlefield b.

Example 1. Consider a simple contest with two players and two battlefields where $c = a = 1$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, $w = (1,1)$, and $\beta = 1$. Suppose that both players allocate all of their resources to battlefield 1, so $x_1 = x_2 = (1,0)$. Then agent 1's success profile is given by $y_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the payoff to agent 1 is given by $\pi_1 = \frac{1}{2}$. However if agent 1 reallocates a small portion ε of her resources from battlefield 1 to battlefield 2 then her success vector will equal $y_1 = \left(\frac{1-\varepsilon}{2-\varepsilon}, 1\right)$ and her payoff will equal $\pi_1 = \left(\frac{1}{2}\left(\frac{2-\varepsilon}{1-\varepsilon}\right) + \frac{1}{2}\right)^{-1}$. Taking the limit as ε converges to zero obtains $\lim_{\varepsilon\to 0} \pi_1 = \frac{2}{3}$.

3.2 Strict Quasiconcavity

Theorem 2 states that the payoff π_i to agent i is strictly quasiconcave over $x_i \in \mathbb{R}^m_{++}$. Since π_i is continuous and quasiconcave over $x_i \in \mathbb{R}^m_{++}$, the first order conditions on agent i's strategy x_i are sufficient for the maximization of π_i over her strategy set X_i.

Theorem 2. *Agent i's payoff π_i is strictly quasiconcave over $x_i \in \mathbb{R}^n_{++}$.*

Proof. Let g_i denote an increasing function of π_i given by

$$g_i = -\frac{\beta^c}{c\pi_i^c} = -\frac{1}{c}\sum_{b=1}^{m} v_b y_{ib}^{-c} \tag{6}$$

Differentiating y_{ib} with respect to x_{ib} yields

$$\begin{aligned}
\frac{\partial y_{ib}}{\partial x_{ib}} &= \frac{\partial}{\partial x_{ib}}\left[\frac{x_{ib}^a}{\sum_{j=1}^n x_{jb}^a}\right] \\
&= \frac{\sum_{j\neq i} x_{jb}^a}{\left(\sum_{j=1}^n x_{jb}^a\right)^2} a x_{ib}^{a-1} \\
&= \frac{a}{x_{ib}}\left(\frac{\sum_{j\neq i} x_{jb}^a}{\sum_{j=1}^n x_{jb}^a}\right)\left(\frac{x_{ib}^a}{\sum_{j=1}^n x_{jb}^a}\right) \\
&= \frac{a\left(1-y_{ib}\right) y_{ib}}{x_{ib}}
\end{aligned}$$

So differentiating g_i with respect to x_i yields

$$\frac{\partial g_i}{\partial x_{ib}} = \frac{\partial g_i}{\partial y_{ib}}\frac{\partial y_{ib}}{\partial x_{ib}} \tag{7}$$

$$= \frac{v_b}{y_{ib}^{1+c}}\frac{a\left(1-y_{ib}\right) y_{ib}}{x_{ib}} \tag{8}$$

$$= \frac{a v_b\left(1-y_{ib}\right)}{y_{ib}^c x_{ib}} \tag{9}$$

Since the numerator of (9) is decreasing in x_{ib} and the denominator is increasing in x_{ib} we have

$$\frac{\partial^2 g_i}{\partial x_{ib}^2} < 0 \tag{10}$$

Since (9) is constant in x_{ih} for all $h \neq b$, the mixed second order partial derivatives are given by

$$\frac{\partial^2 g_i}{\partial x_{ib} \partial x_{ih}} = 0 \tag{11}$$

Thus the matrix of second order partial derivatives is negative definite, so g_i is strictly concave in x_i. Hence π_i is strictly quasiconcave in x_i since g_i is a strictly increasing function of π_i. □

4 The Best Response

Theorem 3 states that agent i's best response never intersects the boundary of her strategy set. Since every best response must lie in the interior of her strategy set and her payoff is strictly quasiconcave over this region, her best response must be unique.

Theorem 3. *For every strategy profile $x \in X$ such that $x_{ib} = 0$ there exits some alternative strategy $x_i' \in X_i$ such that $\pi_i(x_i', x_{-i}) > \pi_i(x)$.*

Proof. Let $x \in X$ such that $x_{ib} = 0$. Now consider the alternative strategy $\hat{x}_i \in X_i$ such that

$$\hat{x}_{ik} = \varepsilon \frac{w_i}{m} + (1 - \varepsilon) x_{ik} \tag{12}$$

If $x_{jb} > 0$ for some $j \neq i$ then $\pi_i(x) = 0 < \pi_i(\hat{x}_i, x_{-i})$. Alternatively, if $x_{jb} = 0$ for all j then $y_{ib}(x) = 1/n < 1 = y_{ib}(\hat{x}_i, x_{-i})$ for all $\varepsilon > 0$. Since $x_{jb} = 0$ for all $j \in N$ there exists at least one battlefield $h \in B$ such that $x_{jh} > 0$ for some $j \neq i$. Then the limiting value of $y_{ih}(\hat{x}_i, x_{-i})$ as ε approaches zero from above is given by

$$\lim_{\varepsilon \downarrow 0} y_{ih}(\hat{x}_i, x_{-i}) = \lim_{\varepsilon \downarrow 0} \frac{x_{ih}}{\sum_{j=1}^{n} x_{jh}^a} = y_{ih}(x) \tag{13}$$

Hence $\pi_i(\hat{x}_i, x_{-i}) > \pi_i(x)$ for some $\varepsilon > 0$ since π_i is continuous over $y_i \in \mathbb{R}^m_{++}$. □

The first order conditions on agent i's strategy are both necessary and sufficient for the maximization of her payoff since it is continuous and quasiconcave over the interior of her strategy set and every best response lies in the interior. These necessary and sufficient first order conditions are provided by Theorem 4.

Theorem 4. *A strategy $x_i \in X_i$ maximizes agent i's payoff π_i if and only if for all battlefields b and k we have*

$$\frac{v_b(1 - y_{ib})}{y^c_{ib} x_{ib}} = \frac{v_k(1 - y_{ik})}{y^c_{ik} x_{ik}} \tag{14}$$

Proof. Suppose that x_i is a best response for agent i. By Theorem 3, x_i must lie in the interior of agent i's strategy set. Hence agent i's marginal benefit from increasing her allocation to battlefield i must equal her marginal benefit from increasing her allocation to battlefield j. By Corollary 1, agent i's payoff is continuous over the interior of her strategy set, so we have

$$\frac{\partial \pi_i}{\partial x_{ib}} = \frac{\partial \pi_i}{\partial x_{ik}} \tag{15}$$

$$\frac{v_b(1 - y_{ib})}{y^c_{ib} x_{ib}} = \frac{v_k(1 - y_{ik})}{y^c_{ik} x_{ik}} \tag{16}$$

Conversely, if x_i satisfies these first order conditions then it must lie in the interior of agent i's strategy set. By Theorem 3 agent i's best response must lie in the interior of her strategy set. By Theorem 2 agent i's payoff π_i is strictly quasiconcave over this region. By Corollary 1 her payoff is continuous in her strategy over this region. Hence the first order conditions on x_i are sufficient for maximization of agent i's payoff over her strategy set. □

5 The Unique Nash Equilibrium

Theorem 5 states that the equilibrium ratio between agent i's allocation x_{ib} to battlefield b and agent j's allocation x_{jb} to battlefield b is identical to the ratio between agent i's endowment w_i and agent $j's$ endowment w_j. Hence every agent allocates the same proportion of her competitive resources to battlefield b in equilibrium.

Theorem 5. *In every pure strategy Nash equilibrium $x_{ib}w_j = x_{jb}w_i$ for every battlefield b and all agents i, j.*

Proof. The first order conditions for the maximization of π_i are given by

$$\frac{v_b\left(1-y_{ib}\right)}{y_{ib}^c x_{ib}} = \frac{v_k\left(1-y_{ik}\right)}{y_{ik}^c x_{ik}} \tag{17}$$

Let $z_{ib} = x_{ib}^a$ and $Z_b = \sum_{j=1}^n z_{ib}$ and $Z_{-ib} = Z_b - z_{ib}$. Then $y_{ib} = z_{ib}/Z_b$ and $1 - y_{ib} = Z_{-ib}/Z_b$ so the first order conditions can be written as

$$\frac{v_b}{x_{ib}}\left(\frac{Z_b}{z_{ib}}\right)^c \frac{Z_{-ib}}{Z_b} = \frac{v_k}{x_{ik}}\left(\frac{Z_k}{z_{ik}}\right)^c \frac{Z_{-ik}}{Z_k} \tag{18}$$

Dividing the first order conditions on x_i by the corresponding first order conditions on x_j obtains

$$\frac{z_{jb}^c x_{jb} Z_{-ib}}{z_{ib}^c x_{ib} Z_{-jb}} = \frac{z_{jk}^c x_{jk} Z_{-ik}}{z_{ik}^c x_{ik} Z_{-jk}} \tag{19}$$

$$\left(\frac{z_{jb}z_{ik}}{z_{ib}z_{jk}}\right)^c \frac{x_{jb}x_{ik}}{x_{ib}x_{jk}} = \frac{Z_{-jb}Z_{-ik}}{Z_{-ib}Z_{-jk}} \tag{20}$$

Now let $Z_{-ijk} = Z_{-ik} - z_{jk}$ so we can write

$$\left(\frac{z_{jb}z_{ik}}{z_{ib}z_{jk}}\right)^c \frac{x_{jb}x_{ik}}{x_{ib}x_{jk}} = \frac{\left(z_{ib}+Z_{-ijb}\right)\left(z_{jk}+Z_{-ijk}\right)}{\left(z_{jb}+Z_{-ijb}\right)\left(z_{ik}+Z_{-ijk}\right)} \tag{21}$$

$$\left(\frac{z_{jb}z_{ik}}{z_{ib}z_{jk}}\right)^c \frac{x_{jb}x_{ik}}{x_{ib}x_{jk}} = \frac{z_{ib}z_{jk} + z_{jk}Z_{-ijb} + z_{ib}Z_{-ijk} + Z_{-ijb}Z_{-ijk}}{z_{jb}z_{ik} + z_{ik}Z_{-ijb} + z_{jb}Z_{-ijk} + Z_{-ijb}Z_{-ijk}} \tag{22}$$

Now suppose for contradiction that

$$\frac{x_{ib}}{x_{ik}} < \frac{x_{jb}}{x_{jk}} \text{ and } \frac{x_{ib}}{x_{ik}} \leq \frac{x_{\ell b}}{x_{\ell k}} \leq \frac{x_{jb}}{x_{jk}} \text{ for all } \ell \tag{23}$$

Now since $z_{ib} = x_{ib}^a$ we can write

$$\begin{aligned} z_{ib} z_{jk} &\leq z_{jb} z_{ik} && (24) \\ z_{ib} z_{\ell k} &\leq z_{ik} z_{\ell b} && (25) \\ z_{jk} z_{\ell b} &\leq z_{jb} z_{\ell k} && (26) \end{aligned}$$

Summing over $\ell \in N \setminus \{i, j\}$ obtains

$$\begin{aligned} z_{ib} z_{jk} &\leq z_{jb} z_{ik} && (27) \\ z_{ib} Z_{-ijk} &\leq z_{ik} Z_{-ijb} && (28) \\ z_{jk} Z_{-ijb} &\leq z_{jb} Z_{-ijk} && (29) \end{aligned}$$

But then we would have

$$\left(\frac{z_{jb} z_{ik}}{z_{ib} z_{jk}}\right)^c \frac{x_{jb} x_{ik}}{x_{ib} x_{jk}} > 1 > \frac{z_{ib} z_{jk} + z_{jk} Z_{-ijb} + z_{ib} Z_{-ijk} + Z_{-ijb} Z_{-ijk}}{z_{jb} z_{ik} + z_{ik} Z_{-ijb} + z_{jb} Z_{-ijk} + Z_{-ijb} Z_{-ijk}} \tag{30}$$

Since this contradicts (22) we must have

$$\frac{x_{ib}}{x_{ik}} = \frac{x_{jb}}{x_{jk}} \tag{31}$$

Hence $x_{ib} x_{jk} = x_{jb} x_{ik}$ and summing over k obtains $x_{ib} w_j = x_{jb} w_i$. □

Theorem 6 states that there is a unique Nash equilibrium allocation profile x^*. Agent i's unique Nash equilibrium allocation x_{ib}^* to battlefield b is proportional to the share parameter v_b of battlefield b.

Theorem 6. *The unique pure strategy Nash equilibrium is given by* $x_{ib}^* = w_i v_b$

Proof. By Theorem 5 we have $x_{ib} w_j = x_{jb} w_i$ so the fraction of prize b awarded to agent i is given by

$$y_{ib} = \frac{x_{ib}^a}{\sum_{j=1}^n x_{jb}^a} = \frac{w_i^a}{\sum_{j=1}^n w_j^a} = \bar{y}_i \tag{32}$$

By Theorem 4 the necessary and sufficient first order conditions on x_i for the maximization of π_i are given by

$$\frac{v_b\left(1-\bar{y}_i\right)}{x_{ib}\bar{y}_i^c} = \frac{v_k\left(1-\bar{y}_i\right)}{x_{ik}\bar{y}_i^c} \tag{33}$$

$$\frac{x_{ik}}{x_{ib}} = \frac{v_k}{v_b} \tag{34}$$

Now since $\sum_{k=1}^{m} x_{ik} = w_i$ and $\sum_{k=1}^{m} v_k = 1$ we have $x_{ib} = w_i v_b$. □

Theorem 6 states that these multi-battle contests have a unique pure strategy Nash equilibrium under arbitrarily decisive contest success functions. In contrast, multi-battle conflicts where agents expend effect at constant marginal cost often have no pure strategy Nash equilibrium if the contest success function is sufficiently decisive. Baye et al. (1994) show that Tullock contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium if the contest success function is sufficiently decisive. Similarly, Arbatskaya and Mialon (2010) show that multi-activity contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium under sufficiently decisive contest success functions. Ewerhart (2017) shows that Nash equilibrium payoffs in contests where agents expend effort at constant marginal cost converge to those of the Mixed strategy Nash equilibrium in the winner-takes-all contest as the contest success function becomes increasingly decisive.

6 Efficiency and Equity

Theorem 7 states that agent i's Nash equilibrium payoff π_i is a function of her endowment w_i and the decisiveness a of the battlefield success function. Although the unique Nash equilibrium strategy profile is insensitive to the decisiveness parameter a, the Nash equilibrium payoffs are shown to exhibit increasing sensitivity to initial endowments under higher decisiveness levels. As the battlefield success function becomes increasingly decisive, the agent with the largest initial endowment earns an increasingly large share of the total payoff. Hence decreasing the decisiveness parameter obtains a more equal equilibrium payoff distribution without distorting the equilibrium behavior.

Theorem 7. *The unique Nash equilibrium payoff to agent i is given by*

$$\pi_i^* = \frac{\beta w_i^a}{\sum_{\ell=1}^n w_\ell^a} \tag{35}$$

Proof. By Theorem 6 the unique Nash equilibrium allocation of agent i to battlefield b can be written as $x_{ib} = w_i v_b$ so we have

$$y_{ib} = \frac{x_{ib}^a}{\sum_{j=1}^n x_{jb}^a} = \frac{w_i^a v_b^a}{\sum_{j=1}^n w_j^a v_b^a} = \frac{w_i^a}{\sum_{j=1}^n w_j^a} = \bar{y}_i \tag{36}$$

Hence $y_{ib} = \bar{y}_i$ for every battlefield b, so the payoff to agent i can be written as

$$\begin{aligned} \pi_i &= \beta \left(\sum_{b=1}^m v_b \bar{y}_i^{-c} \right)^{-\frac{1}{c}} && (37) \\ &= \beta \bar{y}_i \left(\sum_{b=1}^m v_b \right)^{-\frac{1}{c}} && (38) \end{aligned}$$

Now since $\sum_{b=1}^m v_b = 1$ we have

$$\pi_i = \beta \bar{y}_i = \frac{\beta w_i^a}{\sum_{\ell=1}^n w_\ell^a} \tag{39}$$

□

Theorem 8 states that the Nash equilibrium strategy profile maximizes the total payoff to all n agents. Hence the equilibrium strategy profile is Pareto efficient over the set of feasible outcomes, so any non-equilibrium strategy profile that gives agent i a greater payoff than she earns in equilibrium must give some other agent j a lower payoff than she earns in equilibrium.

Theorem 8. *The maximum total payoff to all n agents over all feasible strategy profiles $x \in X$ is given by*

$$\max_{x \in X} \sum_{i=1}^n \pi_i(x) = \beta$$

Proof. Let Y denote the set of all $y \in \mathbb{R}_{+}^{n\times m}$ such that for all battlefields $b \in B$ the sum of all prizes is given by $\sum_{i=1}^{n} y_{ib} = 1$. Hence Y includes all feasible outcomes. Let g_i denote the strictly increasing function of π_i given by

$$g_i = -\frac{\beta^c}{c\pi_i^c} = -\frac{1}{c}\sum_{b=1}^{m} v_b y_{ib}^{-c} \tag{40}$$

For $\theta \in \triangle^{n-1}$ let G_θ denote a weighted sum of all g_i given by

$$G_\theta = \sum_{i=1}^{n} \theta_i g_i = -\frac{1}{c}\sum_{i=1}^{n}\sum_{b=1}^{m} \theta_i v_b y_{ib}^{-c} \tag{41}$$

Hence G_θ is increasing in π_i for each agent i. Now differentiating G_θ with respect to y_{ib} yields

$$\frac{\partial G_\theta}{\partial y_{ib}} = \frac{\theta_i v_b}{y_{ib}^{c+1}} > 0 \tag{42}$$

and twice differentiating G_θ with respect to y_{ib} yields

$$\frac{\partial^2 G_\theta}{\partial y_{ib}^2} = -\left(c+1\right)\frac{\theta_i v_b}{y_{ib}^{c+2}} < 0 \tag{43}$$

The cross partial derivatives are given by $\frac{\partial G}{\partial y_{ib}\partial y_{jb}} = 0$. Hence G_θ is strictly concave over $y_{Nb} = (y_{1b}, \ldots, y_{nb}) \in \mathbb{R}_{++}^{m}$ so the sufficient first order conditions on y_{Nb} for the maximization of G_θ are given by

$$\frac{\theta_i v_b}{y_{ib}^{c+1}} = \frac{\partial G}{\partial y_{ib}} = \frac{\partial G}{\partial y_{jb}} = \frac{\theta_j v_b}{y_{jb}^{c+1}} \tag{44}$$

$$\left(\frac{\theta_i}{\theta_j}\right)^{\frac{1}{c+1}} = \frac{y_{ib}}{y_{jb}} \tag{45}$$

$$y_{ib} = \frac{\theta_i^{\frac{1}{c+1}}}{\sum_{j=1}^{n}\theta_j^{\frac{1}{c+1}}} \tag{46}$$

Thus if y maximizes G_θ over Y then the payoff π_i to agent i under y is given by

$$\pi_i = \frac{\beta\theta_i^{\frac{1}{c+1}}}{\sum_{j=1}^n \theta_j^{\frac{1}{c+1}}} \tag{47}$$

Now if $y \in Y$ maximizes the total payoff $\sum_{i=1}^n \pi_{ib}$ over Y then it is Pareto efficient over Y, so there exists some $\theta \in \triangle^{n-1}$ such that y maximizes G_θ over Y. Then the total payoff is given by

$$\sum_{i=1}^n \pi_i = \frac{\beta \sum_{i=1}^n \theta_i^{\frac{1}{c+1}}}{\sum_{j=1}^n \theta_j^{\frac{1}{c+1}}} = \beta$$

□

Theorem 9 states that agent i can always obtain an above-equilibrium payoff in the two agent case if her opponent employs a non-equilibrium strategy. Hence agent i's equilibrium payoff is also her minimax payoff in the two agent case.

Theorem 9. *If agent j employs a non-equilibrium strategy and $n = 2$ then agent i can obtain an above-equilibrium payoff.*

Proof. Let x_j denote an allocation employed by agent j and suppose that agent i employs the allocation

$$x_{ib} = \frac{w_i x_{jb}}{w_j} \tag{48}$$

Then the share of prize b awarded to agent i is given by

$$y_{ib} = \frac{x_{ib}^a}{x_{ib}^a + x_{jb}^a} = \frac{w_i^a}{w_i^a + w_j^a} = \bar{y}_i \tag{49}$$

Hence the payoff to agent i is given by

$$\pi_i = \beta \left(\sum_{b=1}^m v_b \bar{y}_i^{-c} \right)^{-\frac{1}{c}} = \beta \bar{y}_i \left(\sum_{b=1}^m v_b \right)^{-\frac{1}{c}} = \frac{\beta w_i^a}{w_i^a + w_j^a} \tag{50}$$

Thus by Theorem 7 agent i can always obtain at least her unique Nash equilibrium payoff. Now if $x_{jb} \neq w_j v_b$ then the strategy given by Equation (48) does not satisfy the first order conditions for the maximization of agent i's payoff. Hence it is not a best response by Theorem 4, so there exists some alternative strategy that does even better. $\square$

As illustrated by Example 2, the aggregate total payoff across all n agents depends on the strategies employed. Although the Nash equilibrium strategy profile is Pareto efficient, many other strategy profiles result in Pareto dominated outcomes. Even in the two agent case, this contest is not strictly competitive since both the "size of the pie" and the "division of the pie" depend on the allocation strategy selected by each agent.

Example 2. Consider the simple contest with two players and two battlefields where $c = a = 1$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $w = (1, 1)$. When both agents employ their equilibrium strategies we have $x_{ib} = \frac{1}{2}$ and $y_{ib} = \frac{1}{2}$ for every agent i and every battlefield b. Hence the payoff to each agent is given by $\pi_i = \frac{1}{2}\beta$, so the total payoff to both agents is equal to β. Now suppose instead that agent 1 employs the non-equilibrium strategy $x_1 = \left(\frac{1}{3}.\frac{2}{3}\right)$, and agent 2 employs the non-equilibrium strategy $x_2 = \left(\frac{2}{3}.\frac{1}{3}\right)$. In this case agent 1's success profile is $y_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and agent 2's success profile is $y_2 = \left(\frac{2}{3}.\frac{1}{3}\right)$. Hence the payoff to each agent is given by $\pi_i = \frac{4}{9}\beta$, so the total payoff to both agents is equal to $\frac{8}{9}\beta$.

7 The Two-Stage Multi-Battle Contest

Consider a two-stage multi-battle contest where n agents first make costly investments in rent seeking resources and then allocate these resources between m complementary battlefields. In the first stage, each agent $i \in N$ simultaneously selects her rent seeking expenditure $w_i \in \mathbb{R}_+$ and purchases rent seeking resources at unit cost. In the second stage, agents observes the purchases made by others and then simultaneously allocate their resources over m complementary battlefields. Let $x_{ib} \in \mathbb{R}_+$ denote the quantity of competitive resources that agent i allocates to battlefield b. Agent i's allocation profile $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}_+^m$ must satisfy the budget constraint $\sum_{k=1}^{m} x_{tib} = w_i$. If every agent allocates zero competitive resources to battlefield b, then agent i's share y_{ib} of prize b is given by $y_{ib}(x) = \frac{1}{n}$. Otherwise, agent i's share of prize b is given by equation (2). The total rent obtained by agent i at the end of the second stage is given by

$$R_i(x) = \beta \left(\sum_{b=1}^{m} v_b y_{ib}(x)^{-c} \right)^{-\frac{1}{c}} \tag{51}$$

The parameter $v_b \in \mathbb{R}_+$ denotes the relative value of battlefield b and the parameter $c \in \mathbb{R}_{++}$ denotes the level of complementarity between battlefields. The net profit to agent i is given by $\pi_i(w, x) = R_i(x) - w_i$. Agent i's first stage expenditure w_i is a sunk cost in the second stage, so her equilibrium allocation to battlefield b in the second stage is given by $x_{ib} = v_b w_i$ by Theorem 6. Then by Theorem 7, agent i's indirect payoff function can be written as

$$\pi_i(w) = \frac{\beta w_i^a}{\sum_{j=1}^{n} w_j^a} - w_i \tag{52}$$

If the decisiveness parameter a is less than or equal to $n/(n-1)$, then a pure strategy Nash equilibrium exists under which agent i's expenditure in the first stage is given by $w_i = a\beta(n-1)/n^2$ as shown by Tullock (1980). As noted by Baye et al. (1994), if the decisiveness parameter a is greater than $n/(n-1)$ then the Tullock contest has no pure strategy Nash equilibrium. A characterization of the first stage expenditures in the mixed strategy Nash equilibrium of these high decisiveness cases is provided by Baye et al. (1994) and Ewerhart (2017). Since first stage expenditures are sunk costs, agent i's unique Nash equilibrium allocation over battlefields in the second stage is given by Theorem 6 at every decisiveness level.

8 The Repeated Multi-Battle Contest

Consider a dynamic multi-battle contest where n agents repeatedly allocate rent seeking resources over m complementary battlefields. In each period, agent $i \in N$ is endowed with a unidimensional stock $w_i \in \mathbb{R}_{++}$ of competitive resources. Let $x_{tib} \in \mathbb{R}_+$ denote the quantity of competitive resources that agent i devotes to battlefield b in period t, so $x_{ti} = (x_{ti1}, \ldots, x_{tim}) \in \mathbb{R}_+^m$ must satisfy the budget constraint

$$\sum_{k=1}^{m} x_{tib} = w_i \tag{53}$$

Let $X_i = \left\{x_{ti} \in \mathbb{R}_+^m : \sum_{k=1}^{m} x_{tib} = w_i\right\}$ denote the set of agent i's feasible allocations and let $X = \prod_{i \in N} X_i$ denote the set of all feasible allocation profiles. Let $x_t = (x_{t1}, \ldots, x_{tn}) \in X_t$ denote the allocation profile in period t. When agent i selects her allocation x_{ti} in period t, she can observe the entire history of prior allocations profiles $h_t = (x_0, \ldots, x_{t-1}) \in X^t$. Let $H = \cup_{t \in \mathbb{N}} X^t$ denote the set of all feasible histories. Agent i's strategy $\phi_i : H \to X_i$ gives her allocation in period t as a function of the history h_t such that $x_{ti} = \phi_i (h_t)$ and the allocation profile in period t is given by $x_t = \phi (h_t)$. In each battlefield b, agents compete over a distinct divisible prize. If every agent allocates zero competitive resources to battlefield b, then agent i's share y_{tib} of prize b at time t is given by $y_{tib} (x_t) = \frac{1}{n}$. Otherwise the share of prize b awarded to agent i at time t is given by

$$y_{tib} (x_t) = \frac{x_{tib}^a}{\sum_{j=1}^{n} x_{tjb}^a} \tag{54}$$

Agent i's battlefield success vector in period t is given by $y_{ti} = (y_{ti1}, \ldots, y_{tim}) \in \mathbb{R}_+^m$. Each of the m prizes serves as a complementary input to agent i's period t payoff $\pi_{ti} \in \mathbb{R}_+$, which exhibits constant elasticity of substitution between prizes. If $y_{ti} \notin \mathbb{R}_{++}^m$ then agent i's payoff is given by $\pi_{ti} (y_{ti}) = 0$. For $y_i \in \mathbb{R}_{++}^m$ agent i's period t payoff π_{ti} is given by

$$\pi_{ti} (y_{ti}) = \beta \left(\sum_{b=1}^{m} v_b y_{tib}^{-c} \right)^{-\frac{1}{c}} \tag{55}$$

Let δ denote the rate at which agent i discounts future payoffs. In the finite horizon case, the total discounted payoff to agent i over all T periods is given by

$$\Pi_i (\phi) = \sum_{t=0}^{T} \delta^t \pi_i (y_{tib} (\phi (h_t))) \tag{56}$$

In the infinite horizon case, the total discounted payoff to agent i over all periods is given by

$$\Pi_i\left(\phi\right)=\sum_{t=0}^{\infty}\delta^t\pi_i\left(y_{ti}\left(\phi\left(h_t\right)\right)\right) \tag{57}$$

By Theorem 6 the stage game has a unique Nash equilibrium, so the finitely repeated conflict has a unique subgame perfect Nash equilibrium under which agents employ their stage game Nash equilibrium strategy in every period by backward induction.

Corollary 2. *The finitely repeated conflict has a unique subgame prefect Nash equilibrium under which agents employ their stage game Nash equilibrium strategy in every period.*

In general, the infinitely repeated contest has a multiplicity of subgame perfect Nash equilibria. As shown by Fudenberg and Maskin (1986), every feasible outcome of an infinitely repeated game with above minimax payoffs can be obtained in a subgame perfect Nash equilibrium if the space of individually rational strategies is n-dimensional. However, Theorem 8 and Theorem 9 imply that no feasible outcome of the two-agent multi-battle conflict yields strictly above minimax payoffs to both agents. Accordingly, Theorem 10 states that the infinitely repeated two-agent multi-battle conflict has a unique subgame perfect Nash equilibrium.

Theorem 10. *The infinitely repeated two-agent conflict has a unique subgame perfect Nash equilibrium under which each agent employs her stage game Nash equilibrium strategy in every period.*

Proof. By Theorem 7 if both agents employs their stage game Nash equilibrium strategy in every period then agent i's total discounted payoff is given by

$$\Pi_i^*=\sum_{t=0}^{\infty}\frac{\delta^t\beta w_i^a}{\left(w_i^a+w_j^a\right)}=\frac{\beta w_i^a}{\left(1-\delta\right)\left(w_i^a+w_j^a\right)} \tag{58}$$

Let ϕ denote a subgame perfect Nash equilibrium. By Theorem 8 the maximum stage game sum of payoffs is given by β. Hence the total sum of discounted payoffs over both agents $\Pi_i(\phi) + \Pi_j(\phi)$ is less than or equal to $\beta/(1-\delta) = \Pi_i^* + \Pi_j^*$ under every feasible strategy profile. Now suppose for contradiction that agent j's allocation of competitive resources in period t is not equal to her stage game Nash equilibrium allocation. Then there exists some battlefield b such that $\phi_{jb}(h_t) \neq v_b w_j$, so by Theorem 9 agent i can then obtain a payoff in period t that is above her stage game Nash equilibrium payoff and can obtain at least her stage game Nash equilibrium payoff in every subsequent period. Then agent i's total discounted payoff $\Pi_i(\phi)$ must be strictly greater than Π_i^* by the optimality of ϕ_i in equilibrium. Hence agent j's total discounted payoff $\Pi_j(\phi)$ must be strictly less than Π_j^* since $\Pi_i(\phi)+\Pi_j(\phi) \leq \Pi_i^*+\Pi_j^*$. But this contradicts the optimality of ϕ_j since Π_j^* is agent j's minimax payoff by Theorem 9. Hence agent j's allocation of competitive resources must equal her stage game Nash equilibrium allocation in every period of the infinitely repeated conflict. □

9 Conclusion

The marginal value of greater success in one in one contest often depends on the level of success in other contests. For instance, the marginal revenue earned by a ride hailing from from recruiting additional drivers depends in part on the firm's success in marketing their platform to riders. This paper considers multi-battle contests where n asymmetrically endowed agents allocate resources to compete over m complementary battlefields. Each agent is endowed with a unidimensional stock of competitive resources which they allocate over the m battlefields. In each battlefield, agents compete over a divisible prize with a distinct value. The share of each prize awarded to each agent is given by an arbitrarily decisive contest success function. Prizes serve as constant elasticity inputs to payoffs with an arbitrary degree of complementarity.

This prize value structure covers a wide variety of cases of ranging from Cobb-Douglass to perfect complements. The generality of the value structure is important because, in many applications, an agent's marginal value for a small increase in one prize varies gradually with her share of other prizes. Hiring additional drivers can help a ride hailing firm earn more revenue, but the marginal revenue earned from an additional driver varies gradually with the number of riders.

Greater air superiority can help a military faction more effectively control a contested region, but the marginal effectiveness gained from a small increase in air superiority varies gradually with the faction's ground superiority.

These multi-battle contests are shown to possess a unique Nash equilibrium under arbitrarily decisive contest success functions. Further, the infinitely repeated two-agent contest is shown to possess a unique subgame perfect Nash equilibrium. In contrast, conventional blotto contests and multi-battle contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium when success functions are sufficiently decisive. These results indicate that complementarity between prizes can play an important role in stabilizing strategic behavior.

References

Maria Arbatskaya and Hugo M Mialon. Multi-activity contests. *Economic Theory*, 43(1):23–43, 2010.

Michael R Baye, Dan Kovenock, and Casper G De Vries. The solution to the tullock rent-seeking game when r> 2: Mixed-strategy equilibria and mean dissipation rates. *Public Choice*, 81(3-4):363–380, 1994.

Derek J. Clark and Kai A. Konrad. Asymmetric conflict: Weakest link against best shot. *Journal of Conflict Resolution*, 51(3):457–469, 2007.

Cary Deck, Sudipta Sarangi, and Matt Wiser. Electoral college: A multibattle contest with complementarities. 2017. Working Paper.

John Duffy and Alexander Matros. Stochastic asymmetric Blotto games: An experimental study. *Journal of Economic Behavior & Organization*, 139(C): 88–105, 2017.

Florian Englmaier, Pablo Guillen, Loreto Llorente, Sander Onderstal, and Rupert Sausgruber. The chopstick auction: a study of the exposure problem in multi-unit auctions. *International Journal of Industrial Organization*, 27(2): 286–291, 2009.

Christian Ewerhart. Contests with small noise and the robustness of the all-pay auction. *Games and Economic Behavior*, 105, 2017.

Paul Farris, Gerry Yemen, Virginia Weiler, and Kusum L Ailawadi. Uber pricing strategies and marketing communications. Darden Case No. UVA-M-0871, 2014.

Lawrence Friedman. Game-theory models in the allocation of advertising expenditures. *Operations research*, 6(5):699–709, 1958.

Drew Fudenberg and Eric Maskin. The folk theorem in repeated games with discounting or with incomplete information. *Econometrica*, 54(3):533–554, 1986.

Gian Fulgoni and Andrew Lipsman. Digital game changers: how social media will help usher in the era of mobile and multi-platform campaign-effectiveness measurement. *Journal of Advertising Research*, 54(1):11–16, 2014.

Mark A Hurwitz and Richard E Caves. Persuasion or information? promotion and the shares of brand name and generic pharmaceuticals. *The journal of law and economics*, 31(2):299–320, 1988.

Nan Jia and Kyle Mayer. Complementarity in firms' market and political capabilities: An integrated theoretical perspective. In *Strategy Beyond Markets*, pages 437–470. Emerald Group Publishing Limited, 2016.

Martin Kolmar and Hendrik Rommeswinkel. Contests with group-specific public goods and complementarities in efforts. *Journal of Economic Behavior & Organization*, 89:9–22, 2013.

Dan Kovenock and Brian Roberson. *Conflicts with multiple battlefields*, chapter Conflicts with multiple battlefields. Oxford University Press, 2010.

Dan Kovenock and Brian Roberson. The optimal defense of networks of targets. 2017. ESI Working Papers 17-18.

Dan Kovenock, Sudipta Sarangi, and Matt Wiser. All-pay 2x2 hex: a multibattle contest with complementarities. *International Journal of Game Theory*, 44(3): 571–597, Aug 2015.

Bruce R Pirnie, Alan Vick, Adam Grissom, Karl P Mueller, and David T Orletsky. Beyond close air support. forging a new air-ground partnership. Technical report, Rand Corporation, Santa Monica, CA, 2005.

Birendra K Rai and Rajiv Sarin. Generalized contest success functions. *Economic Theory*, 40(1):139–149, 2009.

Brian Roberson. The colonel blotto game. *Economic Theory*, 29(1):1–24, 2006.

Alex Robson. Multi-Item Contests. ANU Working Papers in Economics and Econometrics 2005-446, Australian National University, College of Business and Economics, School of Economics, 2005.

Tetsuya Saito. How do we get cobb-douglas and leontief functions from ces functions: A lecture note on discrete and continuum differentiated object models. *Journal of Industrial Organization Education*, 6(1), 2012.

Balazs Szentes and Robert W. Rosenthal. Beyond chopsticks: Symmetric equilibria in majority auction games. *Games and Economic Behavior*, 45(2):278 – 295, 2003. Special Issue in Honor of Robert W. Rosenthal.

G. Tullock. Efficient rent-seeking. In J.M. Buchanan, R.D. Tollison, and G. Tullock, editors, *Toward a Theory of the Rent Seeking Society*, pages 97–112. Texas A & M University Press, 1980.

Hirofumi Uzawa. Production functions with constant elasticities of substitution. *The Review of Economic Studies*, 29(4):291–299, 1962.