Multi-battle contests over complementary battlefields

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Abstract

Achieving success in one conflict can often enhance the value of being successful in other conflicts. This paper investigates multi-battle contests where agents allocate competitive resources to compete over multiple divisible prizes. The share of each prize awarded to each agent is given by an arbitrarily decisive contest success function. Prizes serve as constant elasticity inputs with an arbitrary degree of complementarity. Such contests are shown to always possess a unique pure strategy Nash equilibrium. In contrast, conventional contests have no pure strategy Nash equilibrium if success functions are sufficiently decisive. If resources are endogenously obtained though costly investments then equilibrium expenditures are shown to coincide with single-battle endogenous resource contests and equilibrium resource distributions are shown to coincide with multi-battle exogenous resource contests. If success functions are sufficiently decisive then equilibrium expenditures may be stochastic, but the proportion of an agent's resources allocated to each battlefield remains deterministic. These results suggest that complementarity between prizes can play an important role in stabilizing the distribution of competitive resources.

1 Introduction

Success in one conflict often enhances the value of success in other conflicts. Ride hailing firms compete to market their platform to riders and recruit drivers.[1] The marginal revenue from recruiting an additional driver depends in part on the firm's success marketing their platform to riders. Social media platforms compete for both users and advertisers. The marginal revenue from an additional user depends in part on the firm's success in obtaining advertisers.[2] Military factions compete for both air supremacy and ground supremacy. A faction's marginal control over a contested region from additional air supremacy may depend in part on the faction's level of ground supremacy.[3] Pharmaceutical firms compete both to convince doctors of their product's effectiveness and to persuade patients to request it.[4] The marginal revenue from persuading an additional patient depends in part on the firm's success in convincing doctors.

This paper considers multi-battle contests where an arbitrary number of agents compete over an arbitrary number of battlefields. In each battlefield, agents compete over a divisible prize with a distinct value. Each agent is endowed with a unidimensional stock of competitive resources which they allocate between battlefields. The share of each prize awarded to each agent is given by an arbitrarily decisive success function. Prizes serve as constant elasticity inputs with an arbitrary degree of complementarity. Such contests are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive success functions. In contrast, conventional multi-battle conflicts and Blotto games have no pure strategy Nash equilibrium if the contest success function is sufficiently decisive. The unique Nash equilibrium strategy profile is shown to be Pareto efficient over the set of feasible outcomes. Any non-equilibrium strategy profile that gives one agent a greater payoff than she earns in equilibrium will give some other agent a lower payoff than she earns in equilibrium. In the two-agent

[1] Farris et al. (2014) provides details regarding the heavy competition for drivers between ride sharing firms. [2] See Fulgoni and Lipsman (2014) for more on complementarities between user-base and advertisers on social media platforms. [3] See Pirnie et al. (2005) for details regarding complementarity between air supremacy and ground supremacy in military conflicts. [4] See Hurwitz and Caves (1988) for more on rent seeking by pharmaceutical firms.

case, the Nash equilibrium payoffs are also the minimax payoffs. Any deviation from equilibrium by one player can be exploited by the other to obtain an above equilibrium payoff.

Some models of conflict treat victory a binary variable where each agent either achieves total victory or suffers total defeat. Yet in many settings "victory and defeat, although polar opposites, are not binary. There are thousands of points along the scale that delineate degrees of success" as noted by Bartholomees (2008). A military faction's success in a given battlefield may exhibit continuous dependence[5] on territorial gain and casualty levels. A pharmaceutical firm's success in marketing a product may exhibit continuous dependence on the number of doctors and patients persuaded. In the multi-battle contests under consideration, an agent's marginal value for additional success in one battlefield varies nonlinearly with her level of success in each of the other battlefields. The existence of a unique pure strategy Nash equilibrium under arbitrarily decisive success functions is a consequence of this nonlinear dependence. Such nonlinearity is possible because battlefield success functions describe shares of divisible prizes rather than probabilities of obtaining indivisible prizes. If prizes were indivisible then prize shares would be binary variables. Each agent would either obtain the entirety of a prize or none of it. If an agent's probability of obtaining each indivisible prize was independently determined by the resources deployed to the corresponding battlefield, then her expected payoff would be linear in her probability[6] of obtaining each prize.

In some cases, agents may be tasked with allocating a fixed quantity of competitive resources over multiple closely related conflicts. A military commander may be tasked with allocating a fixed quantity of troops over multiple conflict zones. A firm's marketing department may be tasked with allocating a fixed advertising budget over multiple target markets. However, in many applications, the assumption of exogenous endowments with no outside utility or opportunity cost can be overly restrictive. In the short run, a firm's research department might be tasked with allocating a fixed budget over multiple projects, but in the

[5] See Gray et al. (2002), Biddle (2004), and Martel (2011) for more on the measurement of military victory on along multiple continuous dimensions. [6] So her expected payoff would be a multi-linear function of her success profile.

long run, the firm is free adjust their overall research expenditure. If resources are endogenously obtained though costly investments then Nash equilibrium expenditures are shown to coincide with single-battle endogenous resource contests and Nash equilibrium resource distributions are shown to coincide with multi-battle exogenous resource contests. If the battlefield success function is sufficiently decisive then equilibrium expenditure may be stochastic, but the proportion of an agent's resources allocated to each battlefield remains deterministic under arbitrarily decisive battlefield success functions.

The remainder of this paper is organized as follows. Section 2 discusses the related literature. Section 3 formally describes the multi-battle contest under investigation. Section 4 establishes key properties of agent i's best response correspondence. Section 5 establishes the existence of a unique pure strategy Nash equilibrium. Section 6 addresses the equity and efficiency of the unique Nash equilibrium outcome. Section 7 considers the case of endogenous resources where agents make costly investments to obtain competitive resources. Section 8 concludes and discusses important implications of the results. All proofs are provided in Appendix E.

2 Related Literature

A significant potion of the previous literature on multi-battle contests considers prizes that are perfect substitutes. Friedman (1958) considers multi-battle contests where two firms make advertising expenditures to compete over sales in several distinct marketing areas. Robson (2005) investigates two-player multi-item contests between resource constrained agents where prizes are perfect substitutes and contest success functions are probabilistic. Roberson (2006) examines two-player blotto games with deterministic winner-take-all contest success functions. A survey of the multi-battle contest literature is provided by Kovenock and Roberson (2010).

A number of previous works consider specific instances of complementarity in multi-battle conflicts. Englmaier et al. (2009) identify asymmetric equilibria in two-bidder auctions over three items where a single item is has no value by itself and three items are worth no more than two items. Szentes and Rosenthal (2003) identify symmetric mixed strategy equilibria in similar auctions over three items where the marginal value increases for the second item and decreases for the third item. The complementarity in these models has a "chopstick" structure, reflecting the idea that a single chopstick is of little use and three chopsticks is little better than two.

Kolmar and Rommeswinkel (2013) examine contests between teams of agents who exert complementary effort and face linear costs. The complementarity in these both of these models is between effort levels rather than between prizes. Intuitively, this type of complementarity is like the complementarity between the left oar and the right oar when rowing a boat. If you only paddle on one side then you will go in circles. Forward progress is most effectively obtained by paddling on both sides. Analogously, Rai and Sarin (2009) consider contests where each agent makes multiple complementary investments to compete over a single prize.

Skaperdas and Syropoulos (1998) consider single-battle two-agent contests where each agent makes a single investment and the single invisible prize exhibits complementarity with effort levels, such that each agent's valuation for the prize depends own her own effort level. Malueg and Yates (2006) considers single-battle contests with homogeneous success functions where each agent makes a single investment and agents have a common value for the single indivisible prize. Ferrarese (2018) considers a single-battle multi-agent generalization of this contest structure where each agent's valuation for the single divisible prize depends on the prize shares obtained by others. In contrast, the present paper considers a multi-battle multi-agent contest with complementarity between prizes where each agent's marginal value for each prize depends non-linearly on her share of every other prize.

Kovenock et al. (2017) consider two-player multi-battle conflicts with deterministic winner-take-all success functions where prizes are perfect substitutes for the defender and perfect complements for the attacker. The complementarity in this multi-battle contest has a "weakest-link" structure such that the attacker need only win one battle to win the overall contest. Clark and Konrad (2007) consider the case of two firms that compete in multiple simultaneous patent races with probabilistic success functions. These success functions are restricted to have unit decisiveness. Each individual patent has a linearly additive value. Obtaining all of the patents yields an additional monopoly rent. If each patent is secured by some firm, but neither firm secures all of the patents, then they split the monopoly rent evenly. This monopoly rent is the only source of complementarity in this model, otherwise the payoff functions are linear.

Kovenock et al. (2015) consider two-player four-battle contests where the winner of each battle is determined by a Tullock contest. Each player has three possible minimal winning sets consisting of two battlefields each. If a player wins both of the battles in at least one of their winning sets then they win the overall contest. Duffy and Matros (2017) examine two-player probabilistic blotto games where players seek to obtain a majority share of the overall prize value. Deck et al. (2017) investigate two-player multi-battle conflicts with linear effort costs where players seek to obtain a majority share of the overall prize value. The complementarity in both of these models has a majoritarian structure where agents are rewarded for obtaining a majority.

In the present paper, complementarity between prizes has a constant-elasticity structure resulting from from the nonlinear dependence of payoffs on battlefield success levels. It includes a continuum of distinct prize valuation structures ranging from Cobb-Douglass to perfect complements. This payoff nonlinearity can occur because battlefield success functions determine an agent's share of divisible prizes rather than her probability of obtaining indivisible prizes. If prizes were indivisible then prize shares would be binary variables. Either an agent would obtains an entire prize or none of it. If probabilities of obtaining indivisible prizes were independently determined by the resources deployed to corresponding battlefields, then an agent's expected payoff would be a linear in her probability[7] of obtaining each prize.

Several previous works find that conventional blotto games and multi-battle conflicts have no pure strategy Nash equilibrium when success functions are sufficiently decisive. Baye et al. (1994) show that Tullock contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium

[7] Expected payoff would be multi-linear in the vector of probabilities.

if the contest success function is sufficiently decisive. Arbatskaya and Mialon (2010) show that two-player multi-activity contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium if the contest success function is sufficiently decisive.

Ewerhart (2017) shows that Nash equilibrium payoffs in probabilistic contests where agents expend effort at constant marginal cost converge to those of the mixed strategy Nash equilibrium in the deterministic winner-takes-all contest as the success function becomes increasingly decisive. Roberson (2006) notes that conventional blotto games have no pure strategy Nash equilibrium unless one player is strong enough to guarantee complete victory in every battlefield. In contrast, the multi-battle contests considered by this paper are shown to possess a unique pure strategy Nash equilibrium under arbitrarily decisive contest success functions.

3 Complementary Battlefields

Consider a multi-battle conflict where n agents simultaneously allocate limited resources over m complementary battlefields. Let $N = \{1, \ldots, n\}$ denote the set of agents and $B = \{1, \ldots, m\}$ denote the set of battlefields. Agent $i \in N$ is endowed with a unidimensional stock $w_i \in \mathbb{R}_{++}$ of competitive resources. Let $x_{ib} \in \mathbb{R}_+$ denote the quantity of competitive resources that agent i devotes to battlefield b. The strategy $x_i = (x_{i1}, \ldots, x_{im}) \in \mathbb{R}^m_+$ employed by agent i must satisfy the budget constraint $\sum_{k=1}^m x_{ib} = w_i$. Let $X_i = \left\{x_i \in \mathbb{R}^m_+ : \sum_{k=1}^m x_{ib} = w_i\right\}$ denote the set of agent i's allocation strategies and let $X = \prod_{i \in N} X_i$ denote the set of all strategy profiles.

In each battlefield, agents compete over a distinct divisible prize. Agent i's share y_{ib} of prize b is given by $y_{ib} = \gamma_{bi}(x)$ such that $\gamma_b : \mathbb{R}^n \to \triangle_{n-1}$ is continuous, homogeneous of degree zero, and independent[8] from irrelevant alternatives. Agent i's share of prize b is increasing in her allocation to battlefield b and

[8] Independence from irrelevant alternatives means that $\gamma_i(x') = \gamma_i(x) / (1 - \gamma_i(x))$ where $x'_j = 0$ and $x'_k = x_k$ for $k \neq j \neq i$.

decreasing in the allocation x_{jb} of agent $j \neq i$ to battlefield b. Hence[9] if $x_{jb} > 0$ then agent i's share of prize b is given by

$$y_{ib} = \gamma_{bi}(x) = \frac{\mu_i x_{ib}^a}{\sum_{j=1}^n \mu_j x_{jb}^a} \tag{1}$$

where $\mu \in \mathbb{R}^n_{++}$. If zero competitive resources are allocated to battlefield b, then agent i's share y_{ib} of prize b is given by $\gamma_{bi}(x) = \mu_i / \sum_{j=1}^n \mu_j$. The parameter $a \in \mathbb{R}_{++}$ denotes the decisiveness of the battlefield success function. In the limit as $a \to \infty$ the entirety of prize b is awarded to the agent who allocates the most resources to battlefield b. Conversely, in the limit as $a \to 0$ agent i's share of prize b is completely insensitive to the strategy profile. Agent i's battlefield success vector is given by $y_i = (y_{i1}, \ldots, y_{im}) \in \mathbb{R}^m_+$. Each of the m prizes serves as a complementary input to agent i's payoff, which exhibits constant elasticity of substitution between prizes. If $y_i \notin \mathbb{R}^m_{++}$ then agent i's payoff is given by $\pi_i(y_i) = 0$.[10] Otherwise agent i's payoff π_i is given by[11]

$$\pi_i(y_i) = \beta \left(\sum_{b=1}^m v_b y_{ib}^{-c_i} \right)^{-\frac{1}{c_i}} \tag{2}$$

The degree of complementary between battlefields for agent i is given by $c_i \in \mathbb{R}_+$. In the limit as $c_i \to \infty$, all m prizes are perfect complements and agent i's payoff is given by $\pi_i(y_i) = \beta \min\{y_{i1}, \ldots, y_{im}\}$. Conversely, in the limit as $c_i \to 0$, the payoff to agent i takes the Cobb-Douglas form[12] $\pi_i(y_i) = \prod_{b=1}^m y_{ib}^{v_b}$. The share parameter $v_b \in \mathbb{R}_{++}$ denotes the relative value of prize b. The sum of all m share parameters is given by $\sum_{b=1}^m v_b = 1$ without loss of generality since

$$\beta \left(\sum_{b=1}^m \lambda v_b y_{ib}^{-c_i} \right)^{-\frac{1}{c_i}} = \lambda^{-\frac{1}{c}} \beta \left(\sum_{b=1}^m v_b y_{ib}^{-c_i} \right)^{-\frac{1}{c_i}} = \lambda^{-\frac{1}{c_i}} \pi_i \tag{3}$$

[9] See Clark and Riis (1998) for details regarding the necessity of this functional form. [10] As shown by Proposition 1 below, continuity requires that $\pi_i(y_i) = 0$ for all $y_i \notin \mathbb{R}^m_{++}$. [11] See Uzawa (1962) for details regarding the necessity of this functional form. [12] See Saito (2012) for a proof of convergence to the Cobb-Douglas form as c_i approaches zero.

4 The Best Response

Proposition 1 states that an agent's payoff is continuous in her share of each prize. Small changes in agent i's share y_{ib} of prize b produce correspondingly small changes in her overall payoff π_i. By definition, agent i's payoff equals zero whenever $y_i = (y_{i1}, \dots, y_{im}) \notin \mathbb{R}^m_{++}$. Hence agent i's payoff π_i approaches zero as her share y_{ib} of prize b approaches zero since π_i is continuous in y_i over the boundary region $\mathbb{R}^m_+ \setminus \mathbb{R}^m_{++}$.

Proposition 1. *Agent i's payoff π_i is continuous in her success vector $y_i \in \mathbb{R}^m_+$*

Proof. See appendix on page 24. □

If any agent $j \in N$ allocates a non-zero quantity $x_{jb} > 0$ of resources to battlefield b then agent i's share y_{ib} of prize b is continuous in her allocation x_{ib} to battlefield b as given by equation (1). Hence agent i's payoff π_i is continuous in her allocation x_i over the interior of her strategy set $\text{int}(X_i) = X_i \cap \mathbb{R}^m_{++}$. However, as illustrated by Example 1 below, if all n agents allocate zero resources to battlefield b then agent i can obtain the entirety of prize b by reallocating an arbitrarily small portion of her resources to battlefield b.

Example 1. Consider a simple contest with two players and two battlefields where $c_1 = c_2 = a = 1$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, $w = (1, 1)$, and $\beta = 1$. Suppose that both players allocate all of their resources to battlefield 1, so $x_1 = x_2 = (1, 0)$. Then agent 1's success profile is given by $y_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the payoff to agent 1 is given by $\pi_1 = \frac{1}{2}$. If agent 1 reallocates a small portion ε of her resources from battlefield 1 to battlefield 2 then her success vector will equal $y_1' = \left(\frac{1-\varepsilon}{2-\varepsilon}, 1\right)$ and her payoff will equal $\pi_1' = \left(\frac{1}{2}\left(\frac{2-\varepsilon}{1-\varepsilon}\right) + \frac{1}{2}\right)^{-1}$. Taking the limit as ε converges to zero obtains $\lim\limits_{\varepsilon \to 0} \pi_1' = \frac{2}{3} > \frac{1}{2} = \pi_1$.

Proposition 2 states that agent i's payoff π_i is strictly quasiconcave in her allocation x_i over the interior of her strategy set. By Proposition 1 agent i's payoff is continuous in her allocation over this region, so the first order conditions on agent i's allocation are sufficient for the maximization of her payoff over the interior of her strategy set.

Proposition 2. *Agent i's payoff π_i is strictly quasiconcave over $x_i \in \mathbb{R}^n_{++}$.*

Proof. See appendix on page 24. □

Proposition 3 states that every resource allocation on the boundary of agent i's strategy set yields a lower payoff than some other allocation in the interior of her strategy set. Consequently agent i's best responses lie in the interior of her strategy set. By Proposition 2 agent i's payoff π_i is strictly quasiconcave in her allocation over the interior of her strategy set. Hence agent i's best response must always be unique since a convex combination between any two distinct best responses would yield an even larger payoff.

Proposition 3. *For every strategy profile $x \in X$ such that $x_{ib} = 0$ there exits some alternative strategy $x_i' \in X_i$ such that $\pi_i\left(x_i', x_{-i}\right) > \pi_i\left(x\right)$.*

Proof. See appendix on page 24. □

Since agent i's payoff π_i is continuous and quasiconcave over allocations x_i in the interior of her strategy set and her unique best response always lies in the interior of her strategy set, the first order conditions on agent i's allocation are both necessary and sufficient for the maximization of her payoff. These first order conditions are provided by Proposition 4.

Proposition 4. *A strategy $x_i \in X_i$ maximizes agent i's payoff π_i if and only if for all battlefields b and k we have*

$$\frac{v_b\left(1 - y_{ib}\right)}{y_{ib}^{c_i} x_{ib}} = \frac{v_k\left(1 - y_{ik}\right)}{y_{ik}^{c_i} x_{ik}} \tag{4}$$

Proof. See appendix on page 25. □

The first order conditions equalize agent i's marginal benefit from competitive resources across all m battlefields. If her marginal payoff from additional competitive resources in battlefield k was higher than her marginal payoff from additional competitive resources in battlefield b, then agent i could achieve a higher payoff by reallocating resources from battlefield b to battlefield k. Rearranging the first order conditions to isolate the ratio x_{ib}/x_{ik} yields

$$\frac{x_{ib}}{x_{ik}} = \frac{v_b y_{ib}^{-c_i}\left(1 - y_{ib}\right)}{v_k y_{ik}^{-c_i}\left(1 - y_{ik}\right)} \tag{5}$$

Since prizes are net complements, the parameter c_i is positive and the right hand side of equation (5) is decreasing in y_{ib} and increasing in y_{ik}. If agent i is best responding and her share of prize b is larger than her share of prize k then the ratio between her allocation to battlefield b and her allocation to battlefield k must be less than the ratio between the share parameters v_b and v_k. Conversely, if her share of prize b is smaller than her share of prize k then the ratio between her allocation to battlefield b and her allocation to battlefield k must be greater than the ratio between the share parameters v_b and v_k.

5 The Nash Equilibrium

Agent i's payoff π_i is a nonlinear function of her share y_{ib} of each prize b.[13] The marginal value of a small increase in her share of one prize depends on her share of all m prizes. Since prizes are net complements, agent i has a relatively larger marginal value for prizes she has a relatively smaller share of. Complementarity incentivizes agent i to allocate relatively more resources to battlefields where she relatively less successful. Accordingly, Proposition 5 says that agent i receives the same share of each prize in equilibrium.

Proposition 5. *In every pure strategy Nash equilibrium $y_{ib} = y_{ik}$ for every agent i and all battlefields b and k.*

Proof. See appendix on page 25. □

If agent i's share of prize b was larger than her share of prize k then there would be some other agent j whose share of prize k was larger than her share of prize

[13] In contrast, if indivisible prizes were awarded probabilistically then agent i's expected payoff would be linear in her probability of receiving prize b.

b, so the ratio between agent i's allocations to battlefields b and k would be larger than the corresponding ratio for agent j. To the contrary, if both agents are best responding then the first order conditions would imply that the ratio between agent i's allocations to battlefields b and k must be smaller than the corresponding ratio for agent j. Since agent i's obtains an equal share of each prize in equilibrium, equation (5) implies that the ratio between her allocation to battlefield b and her allocation to battlefield k must be equal to the ratio between the share parameters v_b and v_k in equilibrium. Proposition 6 characterizes the unique Nash equilibrium allocation profile under which agent i's allocation to battlefield b is proportional to the share parameter v_b of battlefield b.

Proposition 6. *The unique pure strategy Nash equilibrium is given by* $x^*_{ib} = w_i v_b$

Proof. See appendix on page 26. □

Surprisingly, the Nash equilibrium strategy profile depends on neither the level of complementarity c_i between prizes nor on the decisiveness a of the battlefield success function. Nevertheless, the necessary and sufficient first order conditions (5) indicate that agent i's best response generally depends on both these parameters. By Proposition 5 agent i's share of price b is equal to her share of prize k in equilibrium. Under this specific strategy profile, agent i's best response does not depend on the parameters a and c_i because the first order conditions (5) reduce to

$$\frac{x_{ib}}{x_{ik}} = \frac{v_b}{v_k} \tag{6}$$

When agent i obtains the same share of each prize, the presence of complementarity does not make any particular prize relatively more or less important. Consider a military faction that is competing for control over two complementary regions. If it has more control over the first region than the second region, the presence of complementarity increases the faction's marginal value for control over the second region. The strength of this effect depends on the level of complementarity. However, if the faction has an equal amount of control over both regions, then the presence of complementarity does not make either region relatively more valuable.

In contrast to the unique pure strategy Nash equilibrium identified by Proposition 6, conventional blotto games and multi-battle contests where agents expend effect at constant marginal cost generally have no pure strategy Nash equilibrium if the contest success function is sufficiently decisive. Arbatskaya and Mialon (2010) show that multi-activity contests where agents expend effort at constant marginal cost have no pure strategy Nash equilibrium under sufficiently decisive contest success functions. Roberson (2006) notes that conventional blotto games have no pure strategy Nash equilibrium unless one player is strong enough to guarantee victory in every battlefield.

In the unique Nash equilibrium characterized by Proposition 6, each player allocates the same proportion of her competitive resources to battlefield b. When the contest success function is highly decisive then an agent can almost completely dominate a battlefield by allocating slightly more resources to it than any of her competitors. However, as illustrated by Example 2, the presence of net complementarity between prizes makes such deviations from equilibrium unprofitable.

Example 2. Consider a simple contest with two players and two battlefields such that $\beta = 1$, $w = (1, 1)$, and $v = \left(\frac{1}{4}, \frac{3}{4}\right)$. If each player allocates resources to each battlefield in proportion to the share parameters such that $x_1 = x_2 = v$ then each player obtains half of each prize so the success vectors are given by $y_1 = y_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the payoffs are given by $\pi_1 = \pi_2 = \frac{1}{2}$ so agent 1 is best responding by equation (5). If agent 1 were to reallocate a portion ε of her resources from battlefield 1 to battlefield 2 such that $x_1' = \left(\frac{1}{4} - \varepsilon, \frac{3}{4} + \varepsilon\right)$ then the resulting success vector would satisfy $y_{11}' < y_{12}'$. If this reallocation was optimal then by equation (5) it would satisfy $\frac{x_{11}'}{x_{12}'} > \frac{v_1}{v_2} = \frac{1}{3}$. But this contradicts $x_1' = \left(\frac{1}{4} - \varepsilon, \frac{3}{4} + \varepsilon\right)$ so $\pi_1(x) > \pi_1(x_1', x_2)$. If player 1's payoff was linear in her share of each prize such that $\pi_i = \frac{1}{4}y_{i1} + \frac{3}{4}y_{i2}$ then $\pi_1(x_1', x_2) = \frac{1}{4}y_{11}' + \frac{3}{4}y_{12}' > \frac{1}{2} = \pi_1(x)$ so player 1 would have an incentive to reallocate resources from battlefield 1 to battlefield 2.

6 Efficiency and Equity

Proposition 7 states that agent i's Nash equilibrium payoff π_i is a function of her endowment w_i and the decisiveness a of the battlefield success function. Although the unique Nash equilibrium strategy profile is insensitive to the decisiveness parameter a, the Nash equilibrium payoffs are shown to exhibit increasing sensitivity to initial endowments under higher decisiveness levels. As the battlefield success function becomes increasingly decisive, the agent with the largest initial endowment earns an increasingly large share of the total payoff. Hence decreasing the decisiveness parameter obtains a more equal equilibrium payoff distribution without distorting equilibrium behavior.

Proposition 7. *The unique Nash equilibrium payoff to agent i is given by*

$$\pi_i^* = \frac{\beta \mu_i w_i^a}{\sum_{\ell=1}^n \mu_\ell w_\ell^a} \tag{7}$$

Proof. See appendix on page 26. □

Proposition 8 states that the Nash equilibrium strategy profile maximizes the total payoff to all n agents. Hence the equilibrium strategy profile is Pareto efficient over the set of feasible outcomes, so any non-equilibrium strategy profile that gives agent i a greater payoff than she earns in equilibrium must give some other agent j a lower payoff than she earns in equilibrium.

Proposition 8. *The maximum total payoff to all n agents over all feasible strategy profiles $x \in X$ is given by*

$$\max_{x \in X} \sum_{i=1}^n \pi_i(x) = \beta$$

Proof. See appendix on page 27. □

Proposition 9 states that agent i can always obtain an above-equilibrium payoff in the two agent case if her opponent employs a non-equilibrium strategy. Hence agent i's equilibrium payoff is also her minimax payoff in the two agent case.

Proposition 9. *If agent j employs a non-equilibrium strategy and $n = 2$ then agent i can obtain an above-equilibrium payoff.*

Proof. See appendix on page 27. □

As illustrated by Example 3, the aggregate total payoff across all n agents depends on the strategies employed. Although the Nash equilibrium strategy profile is Pareto efficient, many other strategy profiles result in Pareto dominated outcomes. Even in the two agent case, this contest is not strictly competitive since both the "size of the pie" and the "division of the pie" depend on the allocation strategy selected by each agent.

Example 3. Consider the contest with two players and two battlefields where $c_1 = c_2 = a = 1$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $w = (1, 1)$. When both agents employ their equilibrium strategies we have $x_{ib} = \frac{1}{2}$ and $y_{ib} = \frac{1}{2}$ for every agent i and every battlefield b. Hence the payoff to each agent is given by $\pi_i = \frac{1}{2}\beta$, so the total payoff to both agents is equal to β. Now suppose instead that agent 1 employs the non-equilibrium strategy $x_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and agent 2 employs the non-equilibrium strategy $x_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$. In this case agent 1's success profile is $y_1 = \left(\frac{1}{3}, \frac{2}{3}\right)$ and agent 2's success profile is $y_2 = \left(\frac{2}{3}, \frac{1}{3}\right)$. Hence the payoff to each agent is given by $\pi_i = \frac{4}{9}\beta$, so the total payoff to both agents is equal to $\frac{8}{9}\beta$.

7 Endogenous Resources

Preceding sections consider the case of exogenous endowments where agent i's competitive resources w_i have no outside utility or opportunity cost. In many applications, this assumption may be overly restrictive. In the short run, a firm's research department might be tasked with allocating a fixed budget over multiple projects, but in the long run, the firm is free adjust their overall research expenditure. This section considers the case of endogenous resources where agents purchase competitive resources at unit cost and then allocate them over complementary battlefields.

Let $x_{ib} \in \mathbb{R}_+$ denotes the quantity of competitive resources allocated by agent i to battlefield b. Let w_i denote the total quantity of competitive resources purchased by agent i such that $w_i = \sum_{k=1}^{m} x_{ib}$. If every agent allocates zero competitive resources to battlefield b, then agent i's share y_{ib} of prize b is given by $y_{ib}(x) = \frac{1}{n}$. Otherwise, agent i's share of prize b is given by equation (1). Agent i's net profit is given by $u_i(w, x) = \pi_i(x) - w_i(x)$ where π_i is given by equation (2). This section will first consider the case of observable resource acquisition under which which each agent first observes the quantity of competitive resources purchased by every other agent before deciding how to allocate her own resources between battlefields. Proposition 10 characterizes the unique Nash equilibrium resource allocation policy in this case.

Proposition 10. *Agent i's allocation satisfies $x_{ib} = w_i v_b$ in every Nash equilibrium of the of the multi-battle contest with complementary prizes, endogenous resources, and observable resource acquisition.*

Proof. See appendix on page 28. □

The contest with observable resource acquisition can be described as a two stage game. In the first stage, each agent simultaneously purchases competitive resources. In the second stage, each agent observes the expenditures made by others and decides how to allocate the resources she purchased between battlefields. If there is only a single battlefield then complementarity between prizes is irrelevant and the payoff function reduces to

$$\pi_i(w) = \frac{\beta w_i^a}{\sum_{j=1}^{n} w_j^a} - w_i \tag{8}$$

If the decisiveness parameter a is less than or equal to $n/(n-1)$, then a pure strategy Nash equilibrium for this single-battle contest exists under which agent i's expenditure in the first stage is given by $w_i = a\beta(n-1)/n^2$ as shown by Tullock (1980). By Proposition 7 the indirect payoff function for first stage expenditures in the multi-battle case corresponds to the direct payoff function in the single-battle case as given by equation (8). Intuitively, agent i's first stage expenditure w_i is a sunk cost in the second stage, so Proposition 6 characterizes the subgame starting from the second stage after each agent has observed the expenditures made by others.

In multi-battle contests with unobservable resource acquisition, each agent must decide how to allocate her resources between complementary battlefields before observing the expenditure made by others. Proposition 11 characterizes the pure strategy Nash equilibria for this case.

Proposition 11. *If $a \leq n/(n-1)$ then the unique pure strategy Nash equilibrium of the multi-battle contest with complementary prizes, endogenous resources, and unobservable resource acquisition is given by $w_i = a\beta(n-1)/n^2$ and $x_{ib} = w_i v_b$.*

Proof. See appendix on page 28. □

Intuitively, in a pure strategy Nash equilibrium each agent correctly anticipates the expenditures of every other agent. Agents allocate their competitive resources so as to maximize their expected payoff given their beliefs about the purchases made by others. Since purchasing behavior is deterministic in a pure strategy Nash equilibrium, Proposition 6 characterizes the optimization problem faced by agent i in maximizing her expected payoff given her beliefs. Example 4 illustrates this result in the relatively simple case of two agents and two battlefields.

Example 4. Consider a the contest with endogenous resources, two players, and two battlefields where $c_1 = c_2 = a = 1$, and $v = \left(\frac{1}{2}, \frac{1}{2}\right)$. Suppose that the total quantity of competitive resources purchased by each agent is given by $w_1 = w_2 = \frac{1}{4}\beta$ and the quantity of resources allocated to each battlefield by each player is given by $x_{ib} = \frac{1}{8}\beta$. Then the revenue earned by each agent is given by $\pi_1 = \pi_2 = \frac{1}{2}\beta$ and the net payoff earned by each agent is given by $u_i = \pi_i - w_i = \frac{1}{4}\beta$. Differentiating agent i's revenue π_i with respect to x_{ib} yields $\frac{\partial \pi_i}{\partial x_{ib}} = \frac{\partial \pi_i}{\partial y_{ib}} \frac{\partial y_{ib}}{\partial x_{ib}} = (\frac{\beta}{2})(\frac{2}{\beta}) = 1$. Differentiating agent i's net payoff u_i with respect to x_{ib} yields $\frac{\partial u_i}{\partial x_{ib}} = \frac{\partial \pi_i}{\partial x_{ib}} - \frac{\partial w_i}{\partial x_{ib}} = 1 - 1 = 0$. Since π_i is quasiconcave in x_i these first order conditions are sufficient for Nash equilibrium since.

As noted by Baye et al. (1994), if the decisiveness parameter a is greater than $n/(n-1)$ then the single-battle contest has no pure strategy Nash equilibrium. A characterization for the mixed strategy Nash equilibria of these highly decisive single-battle contests is provided by Baye et al. (1994) and Ewerhart (2015). Proposition 12 extends the results of Proposition 11 to the case of multi-battle contests with complementary battlefields and unobservable resource acquisition.

Proposition 12. *If F is a Nash equilibrium mixed strategy profile for a single-battle endogenous resource contest such that $w_i \sim F_i$ and σ is a mixed strategy profile for a multi-battle contest over complementary battlefields such that $w_i \sim F_i$ and $x_{ib} = w_i v_b$ then σ is a Nash equilibrium strategy profile.*

Proof. See appendix on page 29. □

The strategies described by Proposition 12 are composed of an expenditure policy F_i and an allocation policy $x_i(w_i)$. The unique Nash equilibrium allocation strategy $x_{ib}(w_i) = w_i v_b$ for the corresponding exogenous resource contest is given by Proposition 6. Intuitively, Proposition 12 says that the composition of an equilibrium expenditure policy for a single-battle endogenous resource contest with an equilibrium allocation policy for a multi-battle exogenous resource contest always yields a Nash equilibrium strategy for the corresponding multi-battle endogenous resource contest.

8 Conclusion

In many cases, the marginal value of success in one in one contest can depends on the level of success in other contests. If pharmaceutical firms compete both to convince both doctors and patients of their product's effectiveness, then a firm's marginal revenue from persuading an additional patient may depend on the firm's success in convincing doctors. If military factions compete for both air supremacy and ground supremacy, then the marginal control over a contested region from additional air supremacy may depend on the faction's level of ground supremacy. This paper considers multi-battle conflicts where n agents allocate competitive resources to compete over m complementary battlefields. Each agent's payoff exhibits nonlinear dependence on her success in each battlefield. This nonlinearity can occur because an agent's success in each battlefield describes her share of a divisible prize rather than her probability of obtaining an indivisible prize.

The share of each prize awarded to each agent is given by an arbitrarily decisive contest success function. Prizes serve as constant elasticity inputs to payoffs with an arbitrary degree of complementarity. These multi-battle contests are shown to possess a unique Nash equilibrium under arbitrarily decisive contest success functions. In contrast, conventional multi-battle contests have no pure strategy Nash equilibrium when success functions are sufficiently decisive. The equilibrium strategy profile is shown to be Pareto efficient over the set of feasible outcomes. Any non-equilibrium strategy profile that gives one agent a greater payoff than she earns in equilibrium must give some other agent a lower payoff than she earns in equilibrium. In the two-agent case, an agent's Nash equilibrium payoff is also her minimax payoff, so any deviation from equilibrium by one player can be exploited by the other player to obtain an above equilibrium payoff.

In some cases, agents may be tasked with allocating a fixed quantity of competitive resources over multiple closely related conflicts. For example, a military commander might be tasked with allocating a fixed quantity of troops over multiple conflict zones. However, in many applications, the assumption of exogenous endowments with no outside utility or opportunity cost can be overly restrictive. In the short run, a firm's research department might be tasked with allocating a fixed budget over multiple projects, but in the long run, the firm is free adjust their overall research expenditure. If resources are endogenously obtained though costly investments then Nash equilibrium expenditures are shown to coincide with single-battle endogenous resource contests and Nash equilibrium resource distributions are shown to coincide with multi-battle exogenous resource contests. If the battlefield success function is sufficiently decisive then an agent's Nash equilibrium expenditure may be stochastic, but the proportion of her resources allocated to each battlefield remains deterministic. These results suggest that complementarity between prizes can play an important role in stabilizing the distribution of competitive resources.

A Repeated Contests

Consider a dynamic multi-battle contest where n agents repeatedly allocate rent seeking resources over m complementary battlefields. In each period, agent $i \in N$ is endowed with a unidimensional stock $w_i \in \mathbb{R}_{++}$ of competitive resources. Let $x_{tib} \in \mathbb{R}_+$ denote the quantity of competitive resources that agent i devotes to battlefield b in period t, so $x_{ti} = (x_{ti1}, \ldots, x_{tim}) \in \mathbb{R}_+^m$ must satisfy the budget constraint

$$\sum_{k=1}^{m} x_{tib} = w_i \tag{9}$$

Let $X_i = \left\{x_{ti} \in \mathbb{R}_+^m : \sum_{k=1}^{m} x_{tib} = w_i\right\}$ denote the set of agent i's feasible allocations and let $X = \prod_{i \in N} X_i$ denote the set of all feasible allocation profiles. Let $x_t = (x_{t1}, \ldots, x_{tn}) \in X_t$ denote the allocation profile in period t. When agent i selects her allocation x_{ti} in period t, she can observe the entire history of prior allocations profiles $h_t = (x_0, \ldots, x_{t-1}) \in X^t$. Let $H = \cup_{t \in \mathbb{N}} X^t$ denote the set of all feasible histories. Agent i's strategy $\phi_i : H \to X_i$ gives her allocation in period t as a function of the history h_t such that $x_{ti} = \phi_i (h_t)$ and the allocation profile in period t is given by $x_t = \phi (h_t)$.

In each battlefield b, agents compete over a distinct divisible prize. Agent i's share y_{tib} of prize b at time t is given by $y_{tib} = \gamma_{bi} (x_t)$ as given by equation (1). Agent i's battlefield success vector in period t is given by $y_{ti} = (y_{ti1}, \ldots, y_{tim}) \in \mathbb{R}_+^m$. Each of the m prizes serves as a complementary input to agent i's period t payoff $\pi_{ti} \in \mathbb{R}_+$, which exhibits constant elasticity of substitution between prizes. Agent i's period t payoff π_{ti} is as given by equation (2). Let δ denote the rate at which agent i discounts future payoffs. The total discounted payoff to agent i over all periods is given by

$$\Pi_i (\phi) = \sum_{t=0}^{\infty} \delta^t \pi_i (y_{ti} (\phi (h_t))) \tag{10}$$

By Proposition 6 the stage game has a unique Nash equilibrium, so the finitely repeated conflict has a unique subgame perfect Nash equilibrium under which agents employ their stage game Nash equilibrium strategy in every period by backward induction. In general, the infinitely repeated contest has a multiplicity of subgame perfect Nash equilibria. As shown by Fudenberg and Maskin (1986), every feasible outcome of an infinitely repeated game with above minimax payoffs can be obtained in a subgame perfect Nash equilibrium if the space of individually rational strategies is n-dimensional. However, Proposition 8 and Proposition 9 imply that no feasible outcome of the two-agent multi-battle conflict yields strictly above minimax payoffs to both agents. Accordingly, Proposition 13 states that the infinitely repeated two-agent multi-battle conflict has a unique subgame perfect Nash equilibrium despite not being strictly competitive.

Proposition 13. *The infinitely repeated two-agent conflict has a unique subgame perfect Nash equilibrium under which each agent employs her stage game Nash equilibrium strategy in every period.*

Proof. See appendix on page 29. □

B Asymmetric Valuations

Preceding sections assume symmetric valuations for each prize across agents. This section briefly considers the more general case of asymmetric prize valuations where each agent $i \in N$ has a distinct valuation $v_{ib} \in \mathbb{R}_{++}$ for prize b. For $y_i \in \mathbb{R}^m_{++}$ the payoff to player i is given by

$$\pi_i\left(y_i\right) = \beta\left(\sum_{b=1}^{m} v_{ib} y_{ib}^{-c_i}\right)^{-\frac{1}{c_i}} \tag{11}$$

In this case, the necessary and sufficient first order conditions for Nash equilibrium are given by

$$\frac{v_{ib}\left(1-y_{ib}\right)}{y_{ib}^{c_i} x_{ib}} = \frac{v_{ik}\left(1-y_{ik}\right)}{y_{ik}^{c_i} x_{ik}} \tag{12}$$

The necessity and sufficiency of these first order conditions follows directly from Proposition 4, but the resulting Nash equilibria can differ significantly from those that occur in the case of symmetric prize valuations. Example 5 illustrates this difference in the relatively simple case of two agents and two battlefields. Future research may provide a more complete characterization for the case of asymmetric prize valuations.

Example 5. Consider a contest with two players and two battlefields such that $a = \beta = 1$, $w = c = (1, 1)$. Suppose that players have asymmetric prize valuations v_{ib} such that $v_1 = \left(\frac{2}{3}, \frac{1}{3}\right)$, and $v_2 = \left(\frac{1}{3}, \frac{2}{3}\right)$. Suppose that player 1's allocation is given by $x_1 = \left(\frac{1}{3}\kappa, 1 - \frac{1}{3}\kappa\right)$ and player 2's allocation is given by $x_2 = \left(1 - \frac{1}{3}\kappa, \frac{1}{3}\kappa\right)$ where $\kappa = 2 + 2^{1/3} - 2^{2/3} \in \left(\frac{3}{2}, 2\right)$. So player 1's prize shares are given by $y_1 = x_1$ and player 2's prize shares are given by $y_2 = x_2$. Then

$$\frac{x_{22}}{x_{21}} = \frac{x_{11}}{x_{12}} = \frac{3 - \kappa}{\kappa} = \frac{v_{11}\left(1 - y_{11}\right) y_{11}^{-1}}{v_{12}\left(1 - y_{12}\right) y_{12}^{-1}} = \frac{v_{22}\left(1 - y_{22}\right) y_{22}^{-1}}{v_{21}\left(1 - y_{21}\right) y_{21}^{-1}}$$

so both players are best responding and x is a Nash equilibrium allocation profile. Here player 1 obtains a larger share of prize 1 but player 2 obtains a larger share of prize 2. This contrasts with the case of symmetric prize valuations where player i obtains the same share $y_{ib} = \bar{y}_i$ of each prize in equilibrium.

C Asymmetric Decisiveness

Previous sections consider the case of battlefields that exhibit an identical level of decisiveness a. This section briefly considers the more general case of asymmetric decisiveness where each battlefield $b \in B$ has a distinct decisiveness level a_b. If $x_{jb} > 0$ then player i's share of prize b is given by

$$y_{ib} = \gamma_{bi}\left(x\right) = \frac{\mu_i x_{ib}^{a_b}}{\sum_{j=1}^{n} \mu_j x_{jb}^{a_b}} \tag{13}$$

In this case, the necessary and sufficient first order conditions for Nash equilibrium are given by

$$\frac{a_b v_b \left(1 - y_{ib}\right)}{y_{ib}^{c_i} x_{ib}} = \frac{a_k v_k \left(1 - y_{ik}\right)}{y_{ik}^{c_i} x_{ik}} \tag{14}$$

The necessity and sufficiency of these first order conditions follows from the same argument employed in the proof of Proposition 4. The effect of asymmetric endowments in the relatively simple case of two agents and two battlefields is illustrated by Example 6. Future research may provide a more complete characterization of Nash equilibrium under asymmetric battlefield decisiveness.

Example 6. Consider a contest with two players and two battlefields such that $\beta = 1$, $w_1 = w_2$, $c_1 = c_2$, $v = \left(\frac{3}{5}, \frac{2}{5}\right)$, and $a_1 = 1 < 2 = a_2$ so competitive resources are more decisive in battlefield 2 than in battlefield 1. Suppose that the allocation profile is given by $x_1 = x_2 = \left(\frac{3}{7}, \frac{4}{7}\right)$. Then the prize shares are given by $y_1 = y_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and for $i \in \{1, 2\}$ we have

$$\frac{x_{i1}}{x_{i2}} = \frac{3}{4} = \frac{a_1 v_1 (1 - y_{i1}) \, y_{i1}^{-c_i}}{a_2 v_2 (1 - y_{i2}) \, y_{i2}^{-c_i}}$$

Hence both players are best responding and x is a Nash equilibrium allocation profile. Although $v_1 > v_2$, both players allocate more competitive resources to battlefield 2 than battlefield 1. In contrast, if battlefields have symmetric decisiveness levels then equilibrium strategy profiles reliably allocate more resources to battlefields with more valuable prizes.

D Net Substitutes

Previous sections consider the case of prizes that are net complements, where $c_i \geq 0$ for each agent $i \in N$. This section briefly considers prizes that are net substitutes such that $c_i < 0$ for some agents. Proposition 2 stated that agent i's payoff π_i is quasiconcave in her allocation x_i if prizes are net complements. Conversely, if prizes are net substitutes then her payoff might not be quasiconcave in her allocation and the first order conditions provided by Proposition 4 may fail to characterize agent i's best response.

Example 7. Consider a contest with two players and two battlefields such that $a = \beta = 1$, $w_1 = w_2$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $c_1 = c_2 = -3$ so prizes are net substitutes for both agents. If the allocation profile satisfies the first order conditions given by Proposition 4 then $x_1 = x_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ so the prize shares are given by $y_1 =$

$y_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the payoffs are given by $\pi_1 = \pi_2 = \frac{1}{2}$. If agent 1 were to deviate from x_1 to $x_1' = (1, 0)$ and agent 2 maintains her original strategy $x_2' = x_2$ then agent 1's prize shares would be given by $y_1' = \left(\frac{2}{3}, 0\right)$ and her payoff would be given by $\pi_1' = (\frac{1}{2}(\frac{2}{3})^3)^{\frac{1}{3}} > \frac{1}{2}$. Alternatively, if agent i allocated all of her resources to battlefield 2 then $x_1'' = (0, 1)$ and $\pi_1'' = \pi_1'$. Now $x_1 = \frac{1}{2}x_1' + \frac{1}{2}x_1''$ but $\pi_1 < \pi_1' = \pi_1''$ so agent 1's payoff is not quasiconcave in her allocation and the original allocation profile x is not a Nash equilibrium. Hence the first order conditions are not always sufficient for Nash equilibrium when prizes are net substitutes.

Example 7 illustrates a simple case in which the the presence of net substitutes disrupts the Nash equilibrium predictions characterized by Proposition 6. However, the presence of net substitutes need not always produce this disruption. Example 8 illustrates a simple case in which the Nash equilibrium allocation profile remains unaltered despite the presence of net substitutes. Future research may provide a more complete characterization for Nash equilibrium in the case of net substitutes where $c_i < 0$ for some $i \in N$.

Example 8. Consider a contest with two players and two battlefields such that $a = \beta = 1$, $w_1 = w_2$, $v = \left(\frac{1}{2}, \frac{1}{2}\right)$, and $c_1 = c_2 = -1$ so prizes are net substitutes for both agents. If the allocation profile satisfies the first order conditions given by Proposition 4 then $x_1 = x_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ so the prize shares are given by $y_1 = y_2 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the payoffs are given by $\pi_1 = \pi_2 = \frac{1}{2}$. Now if agent 1 were to deviate from x_1 to some alternative allocation x_1' and agent 2 maintains her original strategy $x_2' = x_2$ then agent 1's prize shares would be given by $y_{11}' = \frac{x_{11}'}{x_{11}'+0.5}$ and $y_{11}' = \frac{x_{12}}{x_{12}+0.5} = \frac{1-x_{11}'}{1.5-x_{11}'}$ and her payoff would satisfy

$$\pi_1' = \frac{1}{2}y_1' + \frac{1}{2}y_2' = \frac{x_{11}'}{1+2x_{11}'} + \frac{1-x_{11}'}{3-2x_{11}'}$$
$$\frac{\partial \pi_1'}{\partial x_{11}'} = \frac{8-16x}{(3+4x-4x^2)^2}$$

Hence x_1 is agent 1's unique best response to x_2 and by symmetry x_2 is agent 2's best response to x_1. Hence the x is a Nash equilibrium allocation profile even though prizes are net substitutes for both agents.

E Proofs

Proof of Proposition 1. If $y_i \in \mathbb{R}^m_{++}$then the payoff to agent i is given by

$$\pi_i\left(y_i\right) = \beta \left(\sum_{b=1}^{m} v_b y_{ib}^{-c_i}\right)^{-\frac{1}{c_i}} = \frac{\beta}{\left(\sum_{b=1}^{m} \frac{v_b}{y_{ib}^{c_i}}\right)^{c_i}} \tag{15}$$

Since $c_i > 0$ the fraction $v_b / y_{ib}^{c_i}$ increases without bound as y_{ib} converges to zero, so the sum $\left(\sum_{b=1}^{m} \frac{v_b}{y_{ib}^{c_i}}\right)^{c_i}$ increases without bound as as y_{ib} converges to zero. Hence $\pi_{ib}\left(y_i\right)$ converges to zero as y_{ib} converges to zero, so agent i's payoff π_i is continuous in her success vector y_i □

Proof of Proposition 2. Let g_i denote an increasing function of π_i given by

$$g_i = -\frac{\beta^{c_i}}{c\pi_i^{c_i}} = -\frac{1}{c_i} \sum_{b=1}^{m} v_b y_{ib}^{-c_i} \tag{16}$$

Differentiating y_{ib} with respect to x_{ib} yields

$$\begin{aligned} \frac{\partial y_{ib}}{\partial x_{ib}} &= \frac{\partial}{\partial x_{ib}} \left[\frac{\mu_i x_{ib}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right] = \frac{\sum_{j \neq i} \mu_j x_{jb}^a}{\left(\sum_{j=1}^{n} \mu_j x_{jb}^a\right)^2} a \mu_i x_{ib}^{a-1} \\ &= \frac{a}{x_{ib}} \left(\frac{\sum_{j \neq i} \mu_j x_{jb}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right) \left(\frac{\mu_i x_{ib}^a}{\sum_{j=1}^{n} \mu_j x_{jb}^a}\right) = \frac{a\left(1 - y_{ib}\right) y_{ib}}{x_{ib}} \end{aligned}$$

So differentiating g_i with respect to x_i yields

$$\frac{\partial g_i}{\partial x_{ib}} = \frac{\partial g_i}{\partial y_{ib}} \frac{\partial y_{ib}}{\partial x_{ib}} = \frac{v_b}{y_{ib}^{1+c_i}} \frac{a\left(1 - y_{ib}\right) y_{ib}}{x_{ib}} = \frac{a v_b\left(1 - y_{ib}\right)}{y_{ib}^{c_i} x_{ib}} \tag{17}$$

Since the numerator of (17) is decreasing in x_{ib} and the denominator is increasing in x_{ib} we have $\frac{\partial^2 g_i}{\partial x_{ib}^2} < 0$. Since (17) is constant in x_{ih} for all $h \neq b$, the mixed second order partial derivatives are given by $\frac{\partial^2 g_i}{\partial x_{ib} \partial x_{ih}} = 0$. Thus the matrix of second order partial derivatives is negative definite, so g_i is strictly concave in x_i. Hence π_i is strictly quasiconcave in x_i since g_i is a strictly increasing function of π_i. □

Proof of Proposition 3. Let $x \in X$ such that $x_{ib} = 0$. Now consider the alternative strategy $\hat{x}_i \in X_i$ such that

$$\hat{x}_{ik} = \varepsilon \frac{w_i}{m} + (1 - \varepsilon) x_{ik} \tag{18}$$

If $x_{jb} > 0$ for some $j \neq i$ then $\pi_i(x) = 0 < \pi_i(\hat{x}_i, x_{-i})$. Alternatively, if $x_{jb} = 0$ for all j then $\gamma_{bi}(x) = \mu_i / \sum_{j=1}^n \mu_j < 1 = y_{ib}(\hat{x}_i, x_{-i})$ for all $\varepsilon > 0$. Since $x_{jb} = 0$ for all $j \in N$ there exists at least one battlefield $h \in B$ such that $x_{jh} > 0$ for some $j \neq i$. Then the limiting value of $\gamma_{hi}(\hat{x}_i, x_{-i})$ as ε approaches zero from above is given by

$$\lim_{\varepsilon \downarrow 0} \gamma_{hi}(\hat{x}_i, x_{-i}) = \lim_{\varepsilon \downarrow 0} \frac{\mu_i x_{ih}}{\sum_{j=1}^n \mu_j x_{jh}^a} = y_{ih}(x) \tag{19}$$

Hence $\pi_i(\hat{x}_i, x_{-i}) > \pi_i(x)$ for some $\varepsilon > 0$ since π_i is continuous over $y_i \in \mathbb{R}^m_{++}$. □

Proof of Proposition 4. Suppose that x_i is a best response for agent i. By Proposition 3, x_i must lie in the interior of agent i's strategy set. Hence agent i's marginal benefit from increasing her allocation to battlefield i must equal her marginal benefit from increasing her allocation to battlefield j. By Proposition 1, agent i's payoff is continuous over the interior of her strategy set, so we have

$$\frac{\partial \pi_i}{\partial x_{ib}} = \frac{\partial \pi_i}{\partial x_{ik}} \tag{20}$$

$$\frac{v_b(1 - y_{ib})}{y_{ib}^{c_i} x_{ib}} = \frac{v_k(1 - y_{ik})}{y_{ik}^{c_i} x_{ik}} \tag{21}$$

Conversely, if x_i satisfies these first order conditions then it must lie in the interior of agent i's strategy set. By Proposition 3 agent i's best response must lie in the interior of her strategy set. By Proposition 2 agent i's payoff π_i is strictly quasiconcave over this region. By Proposition 1 her payoff is continuous in her strategy over this region. Hence the first order conditions on x_i are sufficient for maximization of agent i's payoff over her strategy set. □

Proof of Proposition 5. If $y_{ib} < y_{ik}$ then we have

$$\sum_{\ell \neq i} y_{\ell b} = (1 - y_{ib}) > (1 - y_{ik}) = \sum_{\ell \neq i} y_{\ell k} \tag{22}$$

Hence there exists $j \neq i$ such that $y_{jk} < y_{jb}$ and

$$
\begin{aligned}
y_{ib} y_{jk} &< y_{ik} y_{jb} && (23)\\
\left(\frac{\mu_i x_{ib}^a}{Z_b}\right)\left(\frac{\mu_j x_{jk}^a}{Z_k}\right) &< \left(\frac{\mu_i x_{ik}^a}{Z_k}\right)\left(\frac{\mu_j x_{jb}^a}{Z_b}\right) \quad \text{where } Z_b = \sum_{\ell=1}^{n} \mu_\ell x_{\ell b}^a && (24)\\
x_{ib} x_{jk} &< x_{ik} x_{jb} && (25)
\end{aligned}
$$

By Proposition 4 the first order conditions on x_i can be written as

$$
\frac{x_{ib}}{x_{ik}} = \frac{v_b \left(1 - y_{ib}\right) y_{ib}^{-c_i}}{v_k \left(1 - y_{ik}\right) y_{ik}^{-c_i}} \tag{26}
$$

If x is a Nash equilibrium then by equation (26) we have

$$
\begin{matrix} y_{ib} < y_{ik} \\ y_{jk} < y_{jb} \end{matrix} \implies \frac{x_{ib}}{x_{ik}} > \frac{v_b}{v_k} > \frac{x_{jb}}{x_{jk}} \implies x_{ib} x_{jk} > x_{ik} x_{jb} \tag{27}
$$

But this contradicts equation (25). □

Proof of Proposition 6. By Proposition 5 if x is a Nash equilibrium strategy profile then for every agent i there exists $\bar{y}_i \in [0, 1]$ such that for every battlefield b agent i's share of prize b is given by $y_{ib} = \bar{y}_i$. Hence by Proposition 4 the necessary and sufficient first order conditions on x_i for the maximization of π_i are given by

$$
\begin{aligned}
\frac{v_b \left(1 - \bar{y}_i\right)}{x_{ib} \bar{y}_i^{c_i}} &= \frac{v_k \left(1 - \bar{y}_i\right)}{x_{ik} \bar{y}_i^{c_i}} && (28)\\
\frac{v_b}{v_k} &= \frac{x_{ib}}{x_{ik}} && (29)
\end{aligned}
$$

Hence $x_{ib} v_k = x_{ik} v_b$ and summing over k obtains $x_{ib} = w_i v_b$. □

Proof of Proposition 7. By Proposition 5 if x is a Nash equilibrium strategy profile then for every agent i there exists $\bar{y}_i \in [0, 1]$ such that for every battlefield b agent i's share of prize b is given by $y_{ib} = \bar{y}_i$. Hence the payoff to agent i can be written as

$$
\pi_i = \beta \left(\sum_{b=1}^{m} v_b \bar{y}_i^{-c_i} \right)^{-\frac{1}{c_i}} = \beta \bar{y}_i \left(\sum_{b=1}^{m} v_b \right)^{-\frac{1}{c_i}}
$$

Now since $\sum_{b=1}^{m} v_b = 1$ we have

$$\pi_i = \beta \bar{y}_i = \frac{\beta \mu_i w_i^a}{\sum_{\ell=1}^{n} \mu_\ell w_\ell^a} \tag{30}$$

□

Proof of Proposition 8. Let Y denote the set of all $y \in \mathbb{R}_+^{n\times m}$ such that for all battlefields $b \in B$ the sum of all prize shares is given by $\sum_{i=1}^{n} y_{ib} = 1$. Hence Y includes all feasible outcomes. Let g_i denote a strictly increasing function of π_i given by

$$g_i = -\frac{\beta^{c_i}}{c_i \pi_i^{c_i}} = -\frac{1}{c_i} \sum_{b=1}^{m} v_b y_{ib}^{-c_i} \tag{31}$$

For $\theta \in \triangle^{n-1}$ let G_θ denote a weighted sum of all g_i given by

$$G_\theta = \sum_{i=1}^{n} \theta_i g_i = -\sum_{i=1}^{n} \frac{1}{c_i} \sum_{b=1}^{m} \theta_i v_b y_{ib}^{-c_i} \tag{32}$$

Hence G_θ is increasing in π_i for each agent i. Now differentiating G_θ with respect to y_{ib} yields $\frac{\partial G_\theta}{\partial y_{ib}} = \frac{\theta_i v_b}{y_{ib}^{c_i+1}} > 0$ and twice differentiating G_θ with respect to y_{ib} yields $\frac{\partial^2 G_\theta}{\partial y_{ib}^2} = -(c_i+1)\frac{\theta_i v_b}{y_{ib}^{c_i+2}} < 0$. The cross partial derivatives are given by $\frac{\partial G}{\partial y_{ib} \partial y_{jb}} = 0$. Hence G_θ is strictly concave over $y_{Nb} = (y_{1b}, \ldots, y_{nb}) \in \mathbb{R}_{++}^m$ so the first order conditions on y_{Nb} for the maximization of G_θ are given by

$$\frac{\theta_i v_b}{y_{ib}^{c_i+1}} = \frac{\partial G}{\partial y_{ib}} \quad = \quad \frac{\partial G}{\partial y_{jb}} = \frac{\theta_j v_b}{y_{jb}^{c_j+1}} \tag{33}$$

$$\frac{\theta_i}{\theta_j} = \frac{y_{ib}^{c_i+1}}{y_{jb}^{c_j+1}} \quad \Longrightarrow \quad y_{ib} = \bar{y}_i \tag{34}$$

Thus if y maximizes G_θ over Y then the payoff to agent i satisfies $\pi_i = \beta \bar{y}_i$. Now if $y \in Y$ maximizes the total payoff $\sum_{i=1}^{n} \pi_{ib}$ over Y then it is Pareto efficient over Y, so there exists some $\theta \in \triangle^{n-1}$ such that y maximizes G_θ over Y and the total payoff is given by

$$\sum_{i=1}^{n} \pi_i = \beta \sum_{i=1}^{n} \bar{y}_i = \beta$$

□

Proof of Proposition 9. Let x_j denote an allocation employed by agent j and suppose that agent i employs the allocation

$$x_{ib} = \frac{w_i x_{jb}}{w_j} \tag{35}$$

Then the share of prize b awarded to agent i is given by

$$y_{ib} = \frac{\mu_i x_{ib}^a}{\mu_i x_{ib}^a + \mu_j x_{jb}^a} = \frac{\mu_i w_i^a}{\mu_i w_i^a + \mu_j w_j^a} = \bar{y}_i \tag{36}$$

Hence the payoff to agent i is given by

$$\pi_i = \beta \left(\sum_{b=1}^{m} v_b \bar{y}_i^{-c_i} \right)^{-\frac{1}{c_i}} = \beta \bar{y}_i \left(\sum_{b=1}^{m} v_b \right)^{-\frac{1}{c_i}} = \frac{\beta \mu_i w_i^a}{\mu_i w_i^a + \mu_j w_j^a} \tag{37}$$

Thus by Proposition 7 agent i can always obtain at least her unique Nash equilibrium payoff. Now if $x_{jb} \neq w_j v_b$ then the strategy given by Equation (35) does not satisfy the first order conditions for the maximization of agent i's payoff. Hence it is not a best response by Proposition 4, so there exists some alternative strategy that does better. □

Proof of Proposition 10. Suppose that $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a Nash equilibrium of the the multi-battle contest with complementary prizes, endogenous resources, and observable resource acquisition. Let F_i denote the Nash equilibrium probability distribution for agent i's total purchases such that $w_i \sim F_i(w_i)$. Since resource acquisition is observable, agent i's Nash equilibrium allocation x_i must maximize her expected payoff conditional on the observed expenditures $w = (w_1, \ldots, w_n)$ such that $x_i = \text{argmax } E\{u_i|w\} = \text{argmax } \pi_i(x)$ subject to the resource constraint $\sum_{b=1}^{m} x_{ib} = w_i$. Then by Proposition 6 we must have $x_{ib} = w_i v_b$. □

Proof of Proposition 11. Suppose that $x = (x_1, \ldots, x_n)$ is a puire strategy Nash equilibrium of the the multi-battle contest with complementary prizes, endogenous resources, and unobservable resource acquisition. Then agent i's equilibrium strategy maximizes her net payoff u_i such that

$$u_i(x) = \pi_i(x) - \sum_{b=1}^{n} x_{ib}$$

By Proposition 2 u_i is strictly quasiconcave over the interior of the strategy space, so the necessary and sufficient first order conditions for the optimization of agent i's net payoff are given by

$$\begin{aligned}\frac{\partial u_i}{\partial x_{ib}} &= \frac{\partial \pi_i}{\partial x_{ib}} - 1 = 0\\ \frac{\partial \pi_i}{\partial x_{ib}} &= 1 = \frac{\partial \pi_i}{\partial x_{ik}}\end{aligned}$$

Hence the strategy profile x also satisfies the first order conditions for the Nash equilibrium of the corresponding contest with exogenous resources $w_i = \sum_{b=1}^{m} x_{ib}$. By Proposition 4 these first order conditions are necessary and sufficient, so by Proposition 6 we must have $x_{ib} = w_i v_b$. □

Proof of Proposition 12. Let F_i denote the Nash equilibrium probability distribution for agent $i's$ expenditures in a single battle contest such that $w_i \sim F_i(w_i)$. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ denote a mixed strategy profile for a corresponding multi-battle contest with endogenous resources such that $x_{ib} = w_i v_b$ and $w_i \sim F_i(w_i)$ for all $i \in N$. Then by Proposition 7 agent i's indirect payoff function $u_i(x(w_i, w_{-i}))$ takes the form of a single-battle contest. Since F is a Nash equilibrium expenditure distribution for the single battle contest then for all ω_i in the support of F_i

$$\omega_i \in \underset{w_i \in \mathbb{R}_+}{\operatorname{argmax}} \int u_i(x(\omega_i, w_{-i}))\, dF_{-i}(w_{-i}) \tag{38}$$

By Proposition 10, for all expenditure profiles $w \in \mathbb{R}^n_+$

$$x_i(w_i) \in \underset{x_i \in w_i \triangle^{n-1}}{\operatorname{argmax}}\ u_i(x_i, x_{-i}(w_{-i})) \tag{39}$$

By equations (38) and (39), for all ω in the support of F_i we have

$$x_i(\omega) \in \underset{x_i \in \mathbb{R}^n_+}{\operatorname{argmax}} \int u_i(x_i, x_{-i}(w_{-i}))\, dF_{-i}(w_{-i})$$

□

Proof of Proposition 13. By Proposition 7 if both agents employs their stage game Nash equilibrium strategy in every period then agent i's total discounted payoff is given by

$$\Pi_i^* = \sum_{t=0}^{\infty} \frac{\delta^t \beta \mu_i w_i^a}{\left(\mu_i w_i^a + \mu_j w_j^a\right)} = \frac{\beta \mu_i w_i^a}{(1-\delta)\left(\mu_i w_i^a + \mu_j w_j^a\right)} \tag{40}$$

Let ϕ denote a subgame perfect Nash equilibrium strategy profile. By Proposition 8 the maximum stage game sum of payoffs is given by β. Hence the total sum of discounted payoffs over both agents $\Pi_i(\phi) + \Pi_j(\phi)$ is less than or equal to $\beta/(1-\delta) = \Pi_i^* + \Pi_j^*$ under every feasible strategy profile. Now suppose for contradiction that agent j's allocation of competitive resources in period t is not equal to her stage game Nash equilibrium allocation. Then there exists some battlefield b such that $\phi_{jb}(h_t) \neq v_b w_j$, so by Proposition 9 agent i can then obtain a payoff in period t that is above her stage game Nash equilibrium payoff and can obtain at least her stage game Nash equilibrium payoff in every subsequent period. Then agent i's total discounted payoff $\Pi_i(\phi)$ must be strictly greater than Π_i^* by the optimality of ϕ_i in equilibrium. Hence agent j's total discounted payoff $\Pi_j(\phi)$ must be strictly less than Π_j^* since $\Pi_i(\phi) + \Pi_j(\phi) \leq \Pi_i^* + \Pi_j^*$. But this contradicts the optimality of ϕ_j since Π_j^* is agent j's minimax payoff by Proposition 9. Hence agent j's allocation of competitive resources must equal her stage game Nash equilibrium allocation in every period of the infinitely repeated conflict. □

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