Price Stability in Continuous Asset Markets

Abstract

This paper considers continuous-time markets for rental assets with finite transaction rates. Such markets are shown to possess unique Nash equilibrium price paths. Finite order knowledge of rationality is shown to be insufficient for Nash equilibrium. Even if all investors are rational and each investor knows that all other investors are rational, the price path may exhibit significant deviations from equilibrium. The extent of such deviations is shown to depend on interest rates and investor time preferences. These results suggest that artificially low interest rates may contribute to asset price instability in markets with finite order knowledge of rationality.

1 Continuous Rental Markets

Consider a market populated by a continuum of investors $i \in [0,1]$ holding uniform rental assets. Let $p(t) \in \mathbb{R}_+$ denote the asset price at time $t \in \mathbb{R}_+$. Let $q_i(t) \in \mathbb{R}$ denote the quantity of assets held by investor i at time t. Let $m_i(t) \in \mathbb{R}$ denote the quantity of money held by investor i at time t. Agent i's net wealth at time t is given by

$$w_i(t) = m_i(t) + p(t)\, q_i(t) \tag{1}$$

Here $p(t)\, q_i(t)$ denotes the value of assets held by agent i at time t. Let $r \in \mathbb{R}_{++}$ denote the real interest rate and let $z \in \mathbb{R}_{++}$ denote the net rental rate on assets. Then agent i's net revenue at time t is given by[1]

$$\dot{m}_i(t) = rm_i(t) + zq_i(t) - p(t)\, \dot{q}_i(t) \tag{2}$$

Here $rm_i(t)$ denotes interest income, $zq_i(t)$ denotes rental income, and $p(t)\, \dot{q}_i(t)$ denotes the cost of asset purchases. So agent i's profit flow at time t is given by

$$\pi_i(t) = \dot{w}_i(t) = rm_i(t) + zq_i(t) + \dot{p}(t)\, q_i(t) \tag{3}$$

2 Investment Strategies

Let P denote the set of continuous and right-differentiable price paths $p : \mathbb{R}_+ \to \mathbb{R}_+$ such that $p(0) = p_0 \in \mathbb{R}_{++}$ and $\dot{p}$ is right continuous with left limits. At time t, each agent can observe the history of prices over the time interval $[0,t]$. Let $h(t|p) \subseteq P$ denote the subset of price paths that are consistent with p up to time t such that

$$h(t|p) = \{y \in P : y(s) = p(s) \text{ for all } s \in [0,t]\} \tag{4}$$

[1] We write $\dot{x}(t)$ for the right-derivative of x with respect to t.

Here $h(t|p)$ is informationally equivalent the price history up to time t. Let $\mathcal{H}_t = \{h(t|p) : p \in P\}$ denote the set of all time t price histories. Let $\mathcal{H} = \cup_{t \in \mathbb{R}_+} \mathcal{H}_t$ denote the set of all price histories.

Investors can buy and sell assets continuously subject to the finite transaction rate $\gamma \in \mathbb{R}_{++}$ where agent i's net purchase rate satisfies $\dot{q}_i(t) \in [-\gamma, \gamma] = \mathcal{D}$. Agent i's investment strategy $\phi_i : \mathcal{H} \to \mathcal{D}$ specifies her net purchase rate as a function of the price history such that $\dot{q}_i(t) = \phi_i(h(t|p))$. Let Φ_i denote the set of feasible investment strategies $\phi_i : \mathcal{H} \to \mathcal{D}$ such that $\phi_i(h(t|p))$ is right continuous in t with left limits for all $p \in P$. Let $\Phi = \prod_{i \in [0,1]} \Phi_i$ denote the strategy space.

3 Total Discounted Profits

Let $\bar{p} \in \mathbb{R}$ denote the fundamental value of an asset under the price path $p \in P$. It is given by the present value of the rental cash flows generated by the asset such that

$$\bar{p} = \int_0^\infty e^{-rt} z \, dt = \frac{z}{r} \tag{5}$$

Let $\beta > r$ denote the rate at which investors discount future profits. Let $\alpha = \frac{r}{\beta} \in (0,1)$ denote the alignment between the interest rate r and the discount rate β. When α is close to 1 the interest rate closely approximates the discount rate. Conversely, when α is close to 0 then the interest rate is far lower than the discount rate. Let T denote an exponentially distributed stochastic terminal time such that $\Pr(T \leq t) = 1 - e^{-\beta t}$. Then the expected value of agent i's terminal wealth $w_i(T)$ is given by her initial wealth $w_i(0)$ plus her expected total discounted profit.

$$E\{w_i(T)\} = w_i(0) + \int_0^\infty e^{-\beta t} \pi_i(t) \, dt \tag{6}$$

An investment strategy $\phi_i \in \Phi_i$ is said to be optimal under the price path $p \in P$ if it maximizes agent i's total discounted profits. Let $\Phi_i^*(p)$ denote the set of optimal investment strategies under the price path $p \in P$ such that

$$\begin{aligned}\Phi_i^*(p) = \underset{\phi_i \in \Phi_i}{\text{argmax}} \quad & \int_0^\infty e^{-\beta t} \pi_i(t)\, dt \\ \text{s.t.} \quad & \pi_i(t) = r m_i(t) + z q_i(t) + \dot{p}(t) q_i(t) \\ & \dot{m}_i(t) = r m_i(t) + z q_i(t) - p(t) \dot{q}_i(t) \\ & \dot{q}_i(t) = \phi_i(h(p|t))\end{aligned}$$

Proposition 1. *$\phi_i \in \Phi_i^*(p)$ if and only if $\phi_i(h(t|p)) \in \underset{x \in \mathcal{D}}{\text{argmax}}\, x\, \lambda(t|p)$ for $t \in \mathbb{R}_+$*

$$\textit{where } \lambda(t|p) = \alpha \bar{p} - p(t) + (1-\alpha) E_t\{p(T)\}$$

Proposition 1 characterizes optimal investment strategies under the price path p. Here $\lambda(t|p)$ denotes the difference between the current value of an asset and the current price of an asset. The current value of an asset is a convex combination between the fundamental value $\bar{p}$ and the expected terminal price $p(T)$. Optimal investment strategies involve purchasing assets as rapidly as as possible whenever $\lambda(t|p)$ is positive and selling assets as rapidly as possible whenever $\lambda(t|p)$ is negative. If prices remain constant at the fundamental value such that $p(s) = \bar{p}$ for all $s \geq t$ then $\lambda(t|p) = 0$ and investors have no incentive to trade.

4 Feasible Price Paths

Let $Q(t)$ denote the aggregate quantity of assets demanded by investors at time t such that

$$Q(t) = \int_0^1 q_i(t)\, di \tag{7}$$

The quantity supplied depends on the price such that $Q(t) = f^{-1}(p(t))$ where the inverse supply function $f : \mathbb{R} \to \mathbb{R}_{++}$ is an increasing continuously differentiable bijection with bounded elasticity. Let $p_\phi \in P$ denote the price path generated by the strategy profile $\phi \in \Phi$ such that $p_\phi(0) = f(Q(0))$ and $Q_\phi(0) =$

$\int_0^1 q_i(0)\,di$ where

$$\dot{p}_\phi(t) = \dot{Q}_\phi(t) f'(Q_\phi(t)) \quad (8)$$

$$\dot{Q}_\phi(t) = \int_0^1 \phi_i(h(t|p_\phi))\,di \quad (9)$$

Let $P_f = \{p_\phi : \phi \in \Phi\}$ denote the set of all feasible price paths. Let $\mathcal{H}_{ft} = \{h(p|t) : p \in P_f\}$ denote the set of feasible time t histories. Let $\mathcal{H}_f = \cup_{t\in[0,1]}\mathcal{H}_{ft}$ denote the set of all feasible histories. Let $h_f(t|p) = P_f \cap h(t|p)$ denote the set of feasible price paths that are consistent with p up to time t.

5 Nash Equilibrium

Agent i's expected payoff $\pi_i(\phi)$ under the strategy profile $\phi \in \Phi$ is equal to her expected payoff under the price path generated by ϕ such that $\pi_i(\phi) = \pi_i(\phi_i|p_\phi)$. Let $\Phi_i^* : \Phi_{-i} \rightrightarrows \Phi_i$ denote agent i's best response correspondence such that

$$\Phi_i^*(\phi_{-i}) = \underset{\phi_i\in\Phi_i}{\operatorname{argmax}}\ \pi_i(\phi_i, \phi_{-i}) \quad (10)$$

A strategy profile $\phi \in \Phi$ is a Nash equilibrium if each agent best responds to the others such that $\phi_i \in \Phi_i^*(\phi_{-i})$ for all $i \in [0,1]$. Let $P^* = \{p_\phi \in P_f : \phi \in \Phi^*(\phi)\}$ denote the set of Nash equilibrium price paths

Proposition 2. *The unique Nash equilibrium price path p^* is given by*

$$p^*(t) = \begin{cases} f\left(f^{-1}(p(0) + \operatorname{sgn}(\bar{p} - p(0))\,\gamma t)\right) & \text{if } t < t^* \\ \bar{p} & \text{if } t \geq t^* \end{cases}$$
$$\text{where}\ \ t^* = \gamma^{-1}\left|f^{-1}(\bar{p}) - f^{-1}(p(0))\right|$$

Proposition 2 characterizes the unique Nash equilibrium price path. In equilibrium, investors buy assets as rapidly as possible when the price $p(t)$ is below the fundamental value $\bar{p}$ and sell assets as rapidly as possible when the price is above the fundamental value. Once the price reaches the fundamental value it stays there forever. Let t^* denote the time at which the equilibrium price path reaches the fundamental value such that $t^* = \min\{t \in \mathbb{R}_+ : p^*(t) = \bar{p}\}$. The equilibrium price path is completely determined by the fundamental value $\bar{p} = \frac{z}{r}$ and the transaction rate γ. So the equilibrium price path is independent of the alignment $\alpha = \frac{r}{\beta}$ between the interest rate and investor time preferences.

6 Beliefs

Let $\dot{P}$ denote the set of right continuous functions $\dot{p} : \mathbb{R}_+ \to \mathbb{R}$ with left limits. Let $\mathcal{T}$ denote the topology on P induced by the Skorokhod topology on $\dot{P}$. Let $\mathcal{A}$ denote the Borel σ-algebra generated by $\mathcal{T}$. Let $\mathcal{M}_i$ denote the set of conditional probability measures $\mu_i : \mathcal{A} \times \mathcal{A} \to [0, 1]$ that follow Bayes rule whenever possible such that

$$\mu_i (A \cap B|C) = \mu_i (B|C) \, \mu_i (A|B \cap C) \quad \text{for } A, B, C \in \mathcal{A}$$

Intuitively $\mu_i (A|B)$ denotes agent i's subjective probability of the event $p \in A$ conditional on the event $p \in B$. For all $\phi_i \in \Phi_i$ let $\Phi_{it}(\phi_i)$ denote the set of investment strategies $\varphi_i \in \Phi_i$ that coincide with ϕ_i up to time t such that

$$\Phi_{it}(\phi_i) = \{\varphi_i \in \Phi_i : \varphi_i (h_s) = \phi_i (h_s) \text{ for } h_s \in H_s, s \leq t\}$$

A strategy $\phi_i \in \Phi_i$ is a conditional best response to the belief $\mu_i \in \mathcal{M}_i$ under the history $h_t \in \mathcal{H}_t$ if

$$\phi_i \in \underset{\varphi_i \in \Phi_{it}(\phi_i)}{\operatorname{argmax}} \int_P \pi_i (\varphi|p) \, d\mu_i (p|h_t) \tag{11}$$

Let $\Phi_i^* (\mu_i|h_t)$ denote the set of agent i's conditional best responses to the belief $\mu_i \in \mathcal{M}_i$ under the history $h_t \in \mathcal{H}_t$. A strategy $\phi_i \in \Phi_i$ is a sequential best response to the belief $\mu_i \in \mathcal{M}_i$ if it it a conditional best resonse to μ_i under every feasible history such that $\phi_i \in \Phi_i^* (\mu_i|h)$ for all $h \in \mathcal{H}_f$. Let $\Phi_i^* (\mu_i)$ denote the set of agent i's sequential best responses to the belief $\mu_i \in \mathcal{M}_i$.

Proposition 3. $\phi_i \in \Phi_i^* (\mu_i)$ *if and only if* $\phi_i (h) \in \underset{x\in[-\gamma,\gamma]}{\text{argmax}}\, x\, \lambda_i (h|\mu i)$ *for* $h \in \mathcal{H}_f$

$$\text{where } \lambda_i (h_t|\mu_i) = \int_P \lambda (t|p)\, d\mu_i (p|h_t) \text{ for } h \in \mathcal{H}_{ft}$$

Here $\lambda_i (h|\mu_i)$ denotes agent i's subjective expected net marginal benefit of purchasing an asset under the feasible history $h \in \mathcal{H}_f$ and the belief $\mu_i \in \mathcal{M}_i$. Proposition 3 states that sequential best responses involves purchasing assets as quickly as possible whenever the expected marginal benefit is positive and selling assets as quickly as possible whenever the expected marginal benefit is negative.

7 Rationality

Let $\mathcal{M}_i^0$ denote the set of rational beliefs $\mu_i \in \mathcal{M}_i$ such that $\mu_i (P_f|h (t|p)) = 1$ for all $p \in P_f$. Such beliefs are rational in the sense that they only put positive probability on feasible price paths. Let Φ_i^0 denote the set of all strategies $\phi_i \in \Phi_i$ that sequentially best respond to some rational belief $\mu_i \in \mathcal{M}_i^0$ such that $\phi_i \in \Phi_i^* (\mu_i)$. Let P_0 denote the set of all feasible price paths $p = p_\phi \in P_f$ that are generated by some sequentially rational strategy profile $\phi \in \Phi^0$. Let $h_0 (t|p) = P_0 \cap h (t|p)$ denote the set of rational price paths that are consistent with p up to time t.

For all $n \in \mathbb{N}$ let $\mathcal{M}_i^n \subseteq \mathcal{M}_i^{n-1}$ denote the set of beliefs $\mu_i \in \mathcal{M}_i^{n-1}$ that exhibit n^{th} order knowledge of rationality such that $\mu_i (P_{n-1}|h (t|p)) = 1$ for all $p \in P_{n-1}$. Let Φ_i^n denote the set of strategies $\phi_i \in \Phi_i^{n-1}$ such that $\phi_i \in \Phi_i^* (\mu_i)$ for some $\mu_i \in \mathcal{M}_i^n$. Let P_n denote the set of all price paths that are consistent with n^{th} order knowledge of rationality such that $p = p_\phi \in P_{n-1}$ for some $\phi \in \Phi^n$. Let $h_n (t|p) = P_n \cap h (t|p)$ denote the set of n^{th} order rational price paths that are consistent with p up to time t.

Proposition 4. *For all* $n \in \mathbb{N}_0$ *there exists* $\check{p}_n (\alpha) < \bar{p} < \hat{p}_n (\alpha)$ *such that*

$$h_n (t^*|p^*) \;=\; \left\{p \in h_f (t^*|p^*) : p (t) \in [\check{p}_n (\alpha), \hat{p}_n (\alpha)] \text{ for } t > t^*\right\}$$

Here $\hat{p}_n$ denotes the highest price that can occur under n^{th} order knowledge of rationality following convergence to the fundamental value. Conversely $\check{p}_n$ denotes the lowest price that can occur under n^{th} order knowledge of rationality following convergence to the fundamental value. Proposition 4 states that $\hat{p}_n$ is always strictly greater than the fundamental value and $\check{p}_n$ is always strictly less than the fundamental value. Hence finite order knowledge of rationality is always insufficient for Nash equilibrium since it is always consistent with divergence from the fundamental value following convergence to the fundamental value.

Proposition 5. *$\hat{p}_n(\alpha)$ is decreasing in α and $\check{p}_n(\alpha)$ is increasing in α for all $n \in \mathbb{N}_0$ and $\alpha \in (0,1)$. The limiting values are given by*

$$\lim_{\alpha\to 1} \hat{p}_n(\alpha) = \bar{p} = \lim_{\alpha\to 1} \hat{p}_n(\alpha)$$

$$\lim_{\alpha\to 0} \check{p}_n(\alpha) = 0$$

$$\lim_{\alpha\to 0} \hat{p}_n(\alpha) = \infty$$

Proposition 5 states that under n^{th} order knowledge of rationality, the maximal bubble price $\hat{p}_n$ is decreasing in α and the minimal bubble price $\check{p}_n$ is increasing in α. Here $\alpha = \frac{r}{\beta} \in (0,1)$ denotes the alignment between the interest rate and investor time preferences. When the interest rate is closely aligned with investor time preferences then only small deviations from the fundamental value can occur under n^{th} order knowledge of rationality. Conversely, if the interest rate is much lower than the discount rate then large deviations from the fundamental value can occur under n^{th} order knowledge of rationality.

A Proofs

A.1 Investment Strategies

Definition 1. $\xi\left(t|p\right)=z+\dot{p}\left(s\right)-rp\left(s\right)$

Definition 2. $\lambda\left(t|p\right)=\frac{1}{\beta}E_{t}\left\{\xi\left(s|p\right)\right\}=e^{\beta t}\int_{t}^{\infty}e^{-\beta s}\xi\left(s|p\right)ds$

Definition 3. $\bar{p}=\dfrac{z}{r},\ \alpha=\dfrac{r}{\beta}$

Lemma 1. *Agent i's wealth at time t is given by*

$$w_{i}\left(t\right)=e^{rt}w_{i}\left(0\right)+e^{rt}\int_{0}^{t}e^{-r\ell}\xi\left(\ell|p\right)q_{i}\left(\ell\right)$$

Proof. Since $w_{i}\left(t\right)=m_{i}\left(t\right)+p\left(t\right)q_{i}\left(t\right)$ we have

$$\begin{aligned}\dot{w}_{i}\left(t\right) &= rm_{i}\left(t\right)+zq_{i}\left(t\right)+\dot{p}\left(t\right)q_{i}\left(t\right)\\ &= r\left[w_{i}\left(t\right)-p\left(t\right)q_{i}\left(t\right)\right]+zq_{i}\left(t\right)+\dot{p}\left(t\right)q_{i}\left(t\right)\\ &= rw_{i}\left(t\right)+\left[z+\dot{p}\left(t\right)-rp\left(t\right)\right]q_{i}\left(t\right)\\ &= rw_{i}\left(t\right)+\xi\left(t|p\right)q_{i}\left(t\right)\end{aligned}$$

Hence $w_{i}\left(t\right)$ must satisfy

$$w_{i}\left(t\right)=e^{rt}w_{i}\left(0\right)+e^{rt}\int_{0}^{t}e^{-r\tau}\xi\left(\tau|p\right)q_{i}\left(\tau\right)d\tau$$

□

Lemma 2. *The expected value of agent i's terminal wealth is given by*

$$(\beta - r)\, E\left\{w_i\left(T\right)\right\} = \beta w_i(0) + E\left\{\xi\left(T|p\right) q_i\left(T\right)\right\}$$

Proof. By Lemma 1 we have

$$\begin{aligned}
E\left\{w_i\left(T\right)\right\} &= w_i\left(0\right) E\left\{e^{rT}\right\} + E\left\{e^{rT}\int_0^T e^{-r\tau}\xi\left(\tau|p\right) q_i\left(\tau\right) d\tau\right\} \\
E\left\{w_i\left(T\right)\right\} &= w_i\left(0\right)\beta\int_0^\infty e^{(r-\beta)t}dt + E\left\{e^{rT}\int_0^T e^{-r\tau}\xi\left(\tau|p\right) q_i\left(\tau\right) d\tau\right\} \\
E\left\{w_i\left(T\right)\right\} &= \left(\frac{\beta}{\beta - r}\right) w_i\left(0\right) + E\left\{e^{rT}\int_0^T e^{-r\tau}\xi\left(\tau|p\right) q_i\left(\tau\right) d\tau\right\}
\end{aligned}$$

We can write expectation term as

$$\begin{aligned}
& \beta\int_{t=0}^{t=\infty} e^{(r-\beta)t}\int_{s=0}^{s=t} e^{-rs}\xi\left(s|p\right) q_i\left(s\right) ds\ dt \\
= & \ \beta\int_{s=0}^{s=\infty} e^{-rs}\xi\left(s|p\right) q_i\left(s\right)\int_{t=s}^{t=\infty} e^{(r-\beta)t}dt\ ds \\
= & \ \beta\int_{s=0}^{s=\infty} e^{-rs}\xi\left(s|p\right) q_i\left(s\right)\left[\frac{e^{(r-\beta)s}}{\beta - r}\right] ds \\
= & \left(\frac{\beta}{\beta - r}\right)\int_0^\infty e^{-\beta s}\xi\left(s|p\right) q_i\left(s\right) ds \\
= & \left(\frac{1}{\beta - r}\right) E\left\{\xi\left(T|p\right) q_i\left(T\right)\right\}
\end{aligned}$$

□

Lemma 3. *The expected value of agent i's terminal wealth is given by*

$$(\beta - r)\, E\left\{w_i\,(T)\right\} = \beta w_i(0) + \beta q_i\,(0)\,\lambda\,(0|p) + E\left\{\phi_i\,(h\,(T|p))\,\lambda\,(T|p)\right\}$$

Proof. By Lemma 2 we have

$$(\beta - r)\, E\left\{w_i\,(T)\right\} = \beta w_i(0) + E\left\{\xi\,(T|p)\, q_i\,(T)\right\}\Big]$$

We can write the expectation term as

$$\begin{aligned}
E\left\{\xi\,(T|p)\, q_i\,(T)\right\} &= E\left\{\xi\,(T|p)\left[q_i\,(0) + \int_0^T \dot{q}_i\,(t)\,dt\right]\right\} \\
&= q_i\,(0)\,E\left\{\xi\,(T|p)\right\} + E\left\{\xi\,(T|p)\int_0^T \dot{q}_i\,(t)\,dt\right\} \\
&= \beta q_i\,(0)\,\lambda\,(0|p) + E\left\{\xi\,(T|p)\int_0^T \dot{q}_i\,(t)\,dt\right\} \\
&= \beta q_i\,(0)\,\lambda\,(0|p) + \beta\int_0^\infty e^{-\beta s}\xi\,(s|p)\int_0^s \dot{q}_i\,(t)\,dt\,ds
\end{aligned}$$

We can write the integral term as

$$\begin{aligned}
&\beta\int_{s=0}^{s=\infty}\int_{t=0}^{t=s} e^{-\beta s}\xi\,(s|p)\,\dot{q}_i\,(t)\,dt\,ds \\
=\;&\beta\int_{t=0}^{t=\infty}\int_{s=t}^{s=\infty} e^{-\beta s}\xi\,(s|p)\,\dot{q}_i\,(t)\,dt\,ds \\
=\;&\beta\int_{t=0}^{t=\infty}\dot{q}_i\,(t)\int_{s=t}^{s=\infty} e^{-\beta s}\xi\,(s|p)\,ds\,dt \\
=\;&\beta\int_{t=0}^{t=\infty} e^{-\beta t}\dot{q}_i\,(t)\,e^{\beta t}\int_{s=t}^{s=\infty} e^{-\beta s}\xi\,(s|p)\,ds\,dt \\
=\;&\beta\int_0^\infty e^{-\beta t}\phi_i\,(h\,(t|p))\,\lambda\,(t|p)\;dt \\
=\;&E\left\{\phi_i\,(h\,(t|p))\,\lambda\,(t|p)\right\}
\end{aligned}$$

□

Lemma 4. *$\phi_i \in \Phi_i^*(p)$ if and only if for all $t \in \mathbb{R}_+$*

$$\phi_i(h(t|p)) \in \underset{x\in[-\gamma,\gamma]}{\operatorname{argmax}}\, x\,\lambda(t|p)$$

Proof. By Lemma 3 we have $\phi_i \in \Phi_i^*(p)$ if and only if

$$\begin{aligned}
\phi_i &\in \underset{\varphi\in\Phi_i}{\operatorname{argmax}}\, E\{\varphi_i(h(T_i|p))\,\lambda(T_i|p)\,|p\} \\
\phi_i &\in \underset{\varphi\in\Phi_i}{\operatorname{argmax}} \int_0^\infty e^{-\beta s}\varphi_i(h(t|p))\,\lambda(t|p)\,dt
\end{aligned}$$

For $\varphi_i \in \Phi_i$ then $\varphi_i(h(t|p))$ right-differentiable in t and

$$\varphi_i(h(t|p)) \in [-\gamma,\gamma] \quad \text{for all } p \in P$$

Hence $\phi_i \in \Phi_i^*(p)$ if and only if for all $t \in \mathbb{R}_+$

$$\phi_i(h(t|p)) = \underset{x\in[-\gamma,\gamma]}{\operatorname{argmax}}\, x\,\lambda(t|p)$$

□

Lemma 5. $E_t\{\dot{p}(T)\} = \beta E_t\{p(T)\} - \beta p(t)$

Proof. Since $T \sim \operatorname{Exp}(\beta)$ we have

$$\begin{aligned}
E_t\{p(T)\} &= p(t) + E_t\left\{\int_t^T \dot{p}(x)\,dx\right\} \\
E_t\{p(T)\} &= p(t) + \beta e^{\beta t}\int_{s=t}^{s=\infty} e^{-\beta s}\int_{x=t}^{x=s}\dot{p}(x)\,dx\,ds \\
E_t\{p(T)\} &= p(t) + \beta e^{\beta t}\int_{x=t}^{x=\infty}\dot{p}(x)\int_{s=x}^{s=\infty} e^{-\beta s}ds\,dx \\
E_t\{p(T)\} &= p(t) + \beta e^{\beta t}\int_{x=t}^{x=\infty}\dot{p}(x)\left[\frac{e^{-\beta x}}{\beta}\right]\,dx \\
E_t\{p(T)\} &= p(t) + e^{\beta t}\int_{x=t}^{x=\infty} e^{-\beta x}\dot{p}(x)\,dx \\
E_t\{p(T)\} &= p(t) + \frac{1}{\beta}E_t\{\dot{p}(T)\} \\
E_t\{\dot{p}(T)\} &= \beta E_t\{p(T)\} - \beta p(t)
\end{aligned}$$

□

Lemma 6. $\lambda(t|p) = \alpha\bar{p} - p(t) + (1-\alpha) E_t\{p(T)\}$

Proof. By Lemma 5 we have

$$E_t\{\dot{p}(T)\} = \beta E_t\{p(T)\} - \beta p(t)$$

By the definition of $\lambda(t|p)$ we have

$$\begin{aligned}
\beta\lambda(t|p) &= E_t\{\xi(T|p)\} \\
\beta\lambda(t|p) &= E_t\{z + \dot{p}(T) - rp(T)\} \\
\beta\lambda(t|p) &= z + E_t\{\dot{p}(T)\} - rE_t\{p(T)\} \\
\beta\lambda(t|p) &= z + \beta E_t\{p(T)\} - \beta p(t) - rE_t\{p(T)\} \\
\beta\lambda(t|p) &= z - \beta p(t) + (\beta - r) E_t\{p(T)\} \\
\lambda(t|p) &= \frac{z}{\beta} - p(t) + \left(\frac{\beta - r}{\beta}\right) E_t\{p(T)\} \\
\lambda(t|p) &= \left(\frac{r}{\beta}\right)\frac{z}{r} - p(t) + \left(\frac{\beta - r}{\beta}\right) E_t\{p(T)\} \\
\lambda(t|p) &= \alpha\bar{p} - p(t) + (1-\alpha) E_t\{p(T)\}
\end{aligned}$$

□

A.2 Equilibrium

Lemma 7. $\max_{y \in h_f(t|p)} E_t \{y(T)\} = \int_0^\infty e^{-s} f\left(f^{-1}(p(t)) + \frac{\gamma}{\beta} s\right) ds$

Proof. If $y \in h_f(t|p)$ then $y(t) = p(t)$ so for $s > t$

$$\max_{y \in h_f(t|p)} y(t+s) = f\left(f^{-1}(p(t)) + \gamma s\right)$$

In this case, the expected value of the terminal price is

$$\begin{aligned} E_t\{y(T)\} &= \beta \int_0^\infty e^{-\beta s} y(t+s)\, ds \\ &= \beta \int_0^\infty e^{-\beta s} f\left(f^{-1}(p(t)) + \gamma s\right) ds \\ &= \int_0^\infty e^{-u} f\left(f^{-1}(p(t)) + \frac{\gamma}{\beta} u\right) du \qquad \text{where } u = \beta s \end{aligned}$$

□

Lemma 8. *There exists* $\eta \in \mathbb{R}_{++}$ *such that for all* $a, b \in \mathbb{R}_{++}$

$$\frac{f(b)}{f(a)} < \left(\frac{b}{a}\right)^\eta$$

Proof. Since f is continuously differentiable with bounded elasticity there exists $\eta \in \mathbb{R}_{++}$ such that

$$\begin{aligned} \frac{f'(x)}{f(x)} &< \frac{\eta}{x} \qquad \text{for all } x > 0 \\ \int_a^b \frac{f'(x)}{f(x)} dx &< \int_a^b \frac{\eta}{x} dx \\ \log(f(b)) - \log(f(a)) &< \eta \log(b) - \eta \log(a) \\ \frac{f(b)}{f(a)} &< \left(\frac{b}{a}\right)^\eta \end{aligned}$$

□

Lemma 9. $\lim_{x\to\infty}\int_0^\infty e^{-s}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x}ds=1$

Proof. Since f is increasing and bijective then for all $s\geq 0$

$$\begin{aligned}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x} &\geq 1\\ \int_0^\infty e^{-s}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x}ds &\geq 1\end{aligned}$$

By Lemma 8 for all $a,b\in\mathbb{R}_{++}$ we have

$$\frac{f(b)}{f(a)} < \left(\frac{b}{a}\right)^\eta$$

Hence for all $x>f(0)$ we have

$$\begin{aligned}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x} &= \frac{f\left(f^{-1}(x)+\gamma s\right)}{f\left(f^{-1}(x)\right)}\\ \frac{f\left(f^{-1}(x)+\gamma s\right)}{x} &< \left(\frac{f^{-1}(x)+\gamma s}{f^{-1}(x)}\right)^\eta\\ \frac{f\left(f^{-1}(x)+\gamma s\right)}{x} &< \left(1+\frac{\gamma s}{f^{-1}(x)}\right)^\eta\\ \int_0^\infty e^{-s}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x}ds &< \int_0^\infty e^{-s}\left(1+\frac{\gamma s}{f^{-1}(x)}\right)^\eta ds\end{aligned}$$

Since f is increasing and bijective, $\lim_{x\to\infty}f^{-1}(x)=\infty$ and

$$\lim_{x\to\infty}\int_0^\infty e^{-s}\frac{f\left(f^{-1}(x)+\gamma s\right)}{x}ds = 1$$

□

Lemma 10. *There exists* $\hat{p}_f > \bar{p}$ *such that* $\lambda(t|p) < 0$ *if* $p \in P_f$ *and* $p(t) > \hat{p}_f$.

Proof. By Lemma 6 and Lemma 7 there exists $\lambda_0^+ : \mathbb{R}_{++} \to \mathbb{R}$ such that

$$\begin{aligned}
\lambda_0^+(x) &= \max\{\lambda(t|p) : p(t) = x, p \in P_f\} \\
\lambda_0^+(x) &= \alpha\bar{p} - x + (1-\alpha)\int_0^\infty e^{-t} f\left(f^{-1}(x) + \frac{\gamma}{\beta}t\right) dt
\end{aligned}$$

Dividing by x obtains

$$\frac{\lambda_0^+(x)}{x} = \alpha\frac{\bar{p}}{x} - 1 + (1-\alpha)\int_0^\infty e^{-t}\frac{f\left(f^{-1}(x) + \frac{\gamma}{\beta}t\right)}{x} dt$$

Hence by 9 taking the limit as $x \to \infty$ obtains

$$\lim_{x\to\infty} \lambda_0^+(x) = -\infty$$

□

Lemma 11. *There exists* $\check{p}_f \in (0, \bar{p})$ *such that* $\lambda(t|p) > 0$ *if* $p \in P_f$ *and* $p(t) < \hat{p}_f$.

Proof. By Lemma 6 for all $p \in P_f$

$$\lambda(t|p) = \alpha\bar{p} - p(t) + (1-\alpha)E_t\{p(T)\}$$

Since $p(T) \in \mathbb{R}_{++}$ and $\alpha = \frac{r}{\beta} \in (0,1)$ we can write

$$\lambda(t|p) \geq \alpha\bar{p} - x$$

Hence if $p(t) < \alpha\bar{p} \in (0, \bar{p})$ then

$$\lambda(t|p) \geq \alpha\bar{p} - x \geq 0$$

□

Lemma 12. *If $\phi \in \Phi$ is a Nash equilibrium and $p_\phi(t) > \bar{p}$ then $\dot{Q}_\phi(t) = -\gamma$.*

Proof. By Lemma 10 there exists $\hat{\kappa}_0 > \bar{p}$ such that for all $p \in P_f$

$$p(t) > \hat{\kappa}_0 \implies \lambda(t|p) < 0$$

For all $n \in \mathbb{N}$ let $\hat{\kappa}_n \in \mathbb{R}_{++}$ such that

$$\hat{\kappa}_n = \frac{z}{\beta} \sum_{k=0}^{n-1} \left(\frac{\beta - r}{\beta} \right)^k + \left(\frac{\beta - r}{\beta} \right)^n \hat{\kappa}_0$$

If ϕ is a Nash equilibrium then by Lemma 4

$$p_\phi(t) > \hat{\kappa}_0 \implies \dot{Q}_\phi(t) = -\gamma$$

Hence if $p_\phi(t) \leq \hat{\kappa}_0$ then $E_t\{p_\phi(T)\} \leq \hat{\kappa}_0$ and

$$\begin{aligned} \lambda(t|p) &\leq z - \beta p(t) + (\beta - r)\hat{\kappa}_0 \\ \frac{\lambda(t|p)}{\beta} &\leq \frac{z}{\beta} - p(t) + \left(\frac{\beta - r}{\beta} \right) \hat{\kappa}_0 \\ \lambda(t|p) &\leq \beta[\hat{\kappa}_1 - p(t)] \end{aligned}$$

So if $p_\phi(t) > \hat{\kappa}_1$ then $\dot{Q}_\phi(t) = -\gamma$. Suppose that for $p \in P_f$

$$p_\phi(t) > \hat{\kappa}_n^+ \implies \dot{Q}_\phi(t) = -\gamma$$

Hence if $p_\phi(t) \leq \hat{\kappa}_n$ then $E_t\{p_\phi(T)\} \leq \hat{\kappa}_n$ and

$$\begin{aligned} \lambda(t|p) &\leq z - \beta p(t) + (\beta - r) p_n \\ \lambda(t|p) &\leq \beta[\hat{\kappa}_{n+1} - p(t)] \end{aligned}$$

By induction we have

$$p_\phi(t) > \hat{\kappa}_n \implies \dot{Q}_\phi(t) = -\gamma \quad \text{for all } n \in \mathbb{N}$$

Taking the limit as $n \to \infty$ obtains

$$\lim_{n \to \infty} \hat{\kappa}_n = \frac{z}{\beta} \sum_{k=0}^{\infty} \left(\frac{\beta - r}{\beta} \right)^k = \frac{z}{\beta} \left(\frac{\beta}{r} \right) = \frac{z}{r} = \bar{p}$$

Hence $p_\phi(t) > \bar{p} \implies \dot{Q}_\phi(t) = -\gamma$. □

Lemma 13. *If $\phi \in \Phi$ is a Nash equilibrium and $p_\phi(t) < \bar{p}$ then $\dot{Q}_\phi(t) = \gamma$.*

Proof. By Lemma 11 there exists $\check{\kappa}_0 < \bar{p}$ such that for all $p \in P_f$

$$p(t) < \check{\kappa}_0 \implies \lambda(t|p) > 0$$

For all $n \in \mathbb{N}$ let $\check{\kappa}_n \in \mathbb{R}_{++}$ such that

$$\check{\kappa}_n = \frac{z}{\beta} \sum_{k=0}^{n-1} \left(\frac{\beta - r}{\beta}\right)^k + \left(\frac{\beta - r}{\beta}\right)^n \check{\kappa}_0$$

If ϕ is a Nash equilibrium then by Lemma 4

$$p_\phi(t) < \check{\kappa}_0 \implies \dot{Q}_\phi(t) = \gamma$$

Hence if $p_\phi(t) \geq \check{\kappa}_0$ then $E_t\{p_\phi(T)\} \geq \check{\kappa}_0$ and

$$\begin{aligned} \lambda(t|p) &\geq z - \beta p(t) + (\beta - r)\check{\kappa}_0 \\ \frac{\lambda(t|p)}{\beta} &\geq \frac{z}{\beta} - p(t) + \left(\frac{\beta - r}{\beta}\right)\check{\kappa}_0 \\ \lambda(t|p) &\geq \beta\left[\check{\kappa}_1 - p(t)\right] \end{aligned}$$

So if $p_\phi(t) < \check{\kappa}_1$ then $\dot{Q}_\phi(t) = \gamma$. Suppose that for $p \in P_f$

$$p_\phi(t) < \check{\kappa}_n \implies \dot{Q}_\phi(t) = \gamma$$

Hence if $p_\phi(t) \geq \check{\kappa}_n$ then $E_t\{p_\phi(T)\} \geq \check{\kappa}_n$ and

$$\begin{aligned} \lambda(t|p) &\geq z - \beta p(t) + (\beta - r)\check{\kappa}_n \\ \lambda(t|p) &\geq \beta\left[\check{\kappa}_{n+1} - p(t)\right] \end{aligned}$$

So by induction we have

$$p_\phi(t) < \check{\kappa}_n \implies \dot{Q}_\phi(t) = \gamma \quad \text{for all } n \in \mathbb{N}$$

Taking the limit as $n \to \infty$ obtains

$$\lim_{n\to\infty} \check{\kappa}_n = \frac{z}{\beta} \sum_{k=0}^{\infty} \left(\frac{\beta - r}{\beta}\right)^k = \frac{z}{\beta}\left(\frac{\beta}{r}\right) = \frac{z}{r} = \bar{p}$$

Hence $p_\phi(t) < \bar{p} \implies \dot{Q}_\phi(t) = \gamma$. □

Lemma 14. *If* $\phi \in \Phi$ *is a Nash equilibrium and* $p_\phi(t) = \bar{p}$ *then* $\dot{Q}_\phi(t) = 0$.

Proof. Suppose that ϕ is a Nash equilibrium and $p_\phi(t) = \bar{p}$. Then for all $s \in \mathbb{R}_+$

$$\dot{p}_\phi(s) = f'(Q_\phi(s))\,\dot{Q}(s)$$

Now $\dot{Q}_\phi(t)$ is right-continuous in t since $\phi_i(h(t|p))$ is right-continuous in t and

$$\dot{Q}_\phi(t) = \int_0^1 \phi_i(h(t|p_\phi))\,di$$

If $\dot{Q}(t) > 0$ then there exists $s > t$ such that $\dot{Q}(\ell) > 0$ for all $\ell \in [t, s]$ so

$$p_\phi(s) > \bar{p} \quad \text{and} \quad \dot{p}(s) > 0$$

But this contradicts Lemma 12. If $\dot{Q}(t) > 0$ then there exists $s > t$ such that

$$p_\phi(s) < \bar{p} \quad \text{and} \quad \dot{p}(s) < 0$$

But this contradicts Lemma 13. So we must have $\dot{Q}_\phi(t) = 0$. □

Lemma 15. *The unique Nash equilibrium price path is given by*

$$\dot{p}(t) = \gamma f'\left(f^{-1}(p(t))\right) \operatorname{sgn}(z - rp(t))$$

Proof. Let $\phi \in \Phi$ such that for all $i \in [0,1]$ and all $p \in P_f$

$$\phi_i(h(p|t)) = \gamma \operatorname{sgn}(z - rp(t))$$

Hence $\dot{p}_\phi(t) = \dot{Q}_\phi f'(Q_\phi(t))$ where

$$\dot{Q}_\phi(t) = \int_0^1 \phi_i(h(p_\phi|t)) = \gamma \operatorname{sgn}(z - rp(t))$$

If $p_\phi(t) = \frac{z}{r}$ then $\dot{p}_\phi(t) = 0$ so for all $s \geq t$ we have $p_\phi(s) = \frac{z}{r}$ and

$$\begin{aligned} \xi(s|p_\phi) &= z + \dot{p}_\phi(s) - rp_\phi(s) = 0 \\ \lambda(t|p_\phi) &= E_t\{\xi(t|p_\phi)\} = 0 \end{aligned}$$

If $p(t) > \frac{z}{r}$ then for all $s > t$ we have $p_\phi(s) \geq \frac{z}{r}$ and $\dot{p}_\phi(s) \leq 0$ so

$$\begin{aligned} \xi(s|p_\phi) &= z + \dot{p}_\phi(s) - rp_\phi(s) \leq 0 \\ \lambda(t|p_\phi) &= E_t\{\xi(t|p_\phi)\} \leq 0 \end{aligned}$$

If $p(t) < \frac{z}{r}$ then for all $s > t$ we have $p_\phi(s) \leq \frac{z}{r}$ and $\dot{p}_\phi(s) \geq 0$ so

$$\begin{aligned} \xi(t|p_\phi) &= z + \dot{p}_\phi(s) - rp_\phi(s) \geq 0 \\ \lambda(t|p_\phi) &= E_t\{\xi(t|p_\phi)\} \geq 0 \end{aligned}$$

Hence p_ϕ is an equilibrium price path by Lemma 4. By Lemma 12, Lemma 13, and Lemma 14 it is unique. □

A.3 Beliefs

Definition 4. $\lambda_i\left(h|\mu_i\right)=\int_P \lambda\left(t|p\right)d\mu_i\left(p|h\right)$

Lemma 16. *$\phi_i \in \Phi_i^*\left(\mu_i\right)$ if and only if for all $h \in \mathcal{H}_f$ and $t \in \mathbb{R}_+$*

$$\phi_i\left(h\right) \in \underset{x\in[-\gamma,\gamma]}{\text{argmax}}\, x\,\lambda_i\left(h|\mu_i\right)$$

Proof. By definition $\phi_i \in \Phi_i^*\left(\mu_i\right)$ if and only if for all $t \in \mathbb{R}_+$ and $h_t \in \mathcal{H}_{ft}$

$$\phi_i \in \underset{\varphi_i\in\Phi_{it}^C(\phi_i)}{\text{argmax}} \int_P \pi_i\left(\varphi_i|p\right)d\mu_i\left(p|h_t\right)$$

By Lemma 3 this is equivalent to

$$\phi_i \quad \in \quad \underset{\varphi_i\in\Phi_{it}^C(\phi_i)}{\text{argmax}} \int_P E\left\{\varphi_i\left(h\left(T|p\right)\right)\lambda\left(T|p\right)\right\}d\mu_i\left(p|h_t\right)$$

Since $T \sim \text{Exp}\left(\beta\right)$ this means

$$\phi_i \quad \in \quad \underset{\varphi_i\in\Phi_{it}^C(\phi_i)}{\text{argmax}}\, \beta e^{\beta t}\int_0^\infty e^{-\beta s}\int_P \varphi_i\left(h\left(s|p\right)\right)\lambda\left(s|p\right)d\mu_i\left(p|h_t\right)ds$$

Equivalently for all $s \geq t$

$$\phi_i \quad \in \quad \underset{\varphi_i\in\Phi_{it}^C(\phi_i)}{\text{argmax}} \int_P \varphi_i\left(h\left(s|p\right)\right)\lambda\left(s|p\right)d\mu_i\left(p|h_t\right)$$

By the law of iterated expectations this is equivalent to

$$\phi_i \quad \in \quad \underset{\varphi_i\in\Phi_{it}^C(\phi_i)}{\text{argmax}} \int_P \varphi_i\left(h(s|p)\right)\int_P \lambda\left(s|y\right)d\mu_i\left(y|h(s|p)\right)d\mu_i\left(p|h_t\right)$$

$$\phi_i \quad \in \quad \underset{\varphi_i\in\Phi_{it}^C(\phi_i)}{\text{argmax}} \int_P \varphi_i\left(h(s|p)\right)\lambda_i\left(h(s|p)|\mu_i\right)d\mu_i\left(p|h_t\right)$$

Equivalently for all $t \in \mathbb{R}_+$ and $h_t \in \mathcal{H}_{tf}$

$$\phi_i\left(h_t\right) \quad \in \quad \underset{x\in[-\gamma,\gamma]}{\text{argmax}}\, x\,\lambda_i\left(h_t|\mu_i\right)$$

□

A.4 Rationality

Definition 5. For $x \in \mathbb{R}_+$, $c \in [-\gamma, \gamma]$, $a \in (0,1)$, $t \in \mathbb{R}_+$ let

$$g_f(x,c,t) = f\left(f^{-1}(x) + ct\right)$$
$$\lambda_0(x,c,a) = a\bar{p} - x + (1-a)\,E\left\{g_f(x,c,T)\right\}$$

Lemma 17. *For $a \in (0,1)$ there exists $\check{p}_0(a), \hat{p}_0(a) \in \mathbb{R}_{++}$ such that*

$$\check{p}_0(a) < \bar{p} < \hat{p}_0(a)$$
$$\hat{p}_0(a) = \inf\left\{x > \bar{p} : \lambda_0(x,\gamma,a) < 0\right\}$$
$$\check{p}_0(a) = \sup\left\{x < \bar{p} : \lambda_0(x,-\gamma,a) > 0\right\}$$

Proof. By the definition of g_f we have

$$g_f(x,c,t) = f\left(f^{-1}(x) + ct\right)$$
$$E\left\{g_f(x,c,T)\right\} = \beta \int_0^\infty e^{-\beta t} f\left(f^{-1}(x) + ct\right) dt$$
$$E\left\{g_f(x,\gamma,T)\right\} > x$$
$$E\left\{g_f(x,-\gamma,T)\right\} < x$$

By the definition of λ_0 we can write

$$\lambda_0(x,c,a) = a\bar{p} - x + (1-a)\,E\left\{g_f(x,c,T)\right\}$$
$$\lambda_0(\bar{p},c,a) = (1-a)\,E\left\{g_f(\bar{p},c,T) - \bar{p}\right\}$$
$$\lambda_0(\bar{p},\gamma,a) > 0 \quad \text{since } E\left\{g_f(\bar{p},\gamma,T)\right\} > \bar{p}$$
$$\lambda_0(\bar{p},-\gamma,a) < 0 \quad \text{since } E\left\{g_f(\bar{p},-\gamma,T)\right\} < \bar{p}$$

Since $\lambda_0(x,c,a)$ is continuous in x there exists $k > 0$ such that

$$\lambda_0(\bar{p} - \varepsilon, -\gamma, a) < 0 < \lambda_0(\bar{p} + \varepsilon, \gamma, a) \qquad \text{for all } \varepsilon \in (0,k)$$

Hence by Lemma 10 and Lemma 11 there exists $0 < \check{p}_0(a) < \bar{p} < \hat{p}_0(a)$ such that

$$\hat{p}_0(a) = \inf\left\{x > \bar{p} : \lambda_0(x,\gamma,a) < 0\right\}$$
$$\check{p}_0(a) = \sup\left\{x < \bar{p} : \lambda_0(x,-\gamma,a) > 0\right\}$$

□

Definition 6. For $a \in (0,1)$ let $h_0^{\triangleleft}(a) \subseteq P_f$ such that

$$h_0^{\triangleleft}(a) = \{p \in p_f(t^*|p^*) : p(t) \in [\check{p}_0(a), \hat{p}_0(a)] \text{ for } t \geq t^*\}$$

Lemma 18. *If* $p \in P_f$ *then* $\lambda_0(p(s), \gamma, \alpha) = \max_{p \in h_f(s|p)} \lambda(s|p)$

Proof. If $p \in P_f$ then let $p^+ \in h_f(s|p)$ such that

$$p^+(t) = \begin{cases} p(t) & \text{if } t < s \\ f\left(f^{-1}(p(s)) + \gamma[t-s]\right) & \text{if } t \geq s \end{cases}$$

Hence for all $y \in h_f(s|p)$

$$\begin{aligned} p^+(t) &\geq y(t) \\ E_s\left\{p^+(T)\right\} &\geq E_s\{y(T)\} \\ \alpha\bar{p} - p(s) + (1-\alpha) E_s\left\{p^+(T)\right\} &\geq \alpha\bar{p} - p(s) + (1-\alpha) E_s\{y(T)\} \end{aligned}$$

So $\lambda(s|p^+) \geq \lambda(s|y)$ by Lemma 6. By the definition of g_f we have

$$\begin{aligned} g_f(p(s), \gamma, t-s) &= p^+(t) \quad \text{for } t \geq s \\ E\{g_f(p(s), \gamma, T)\} &= E_s\left\{p^+(T)\right\} \\ \lambda_0(p(s), \gamma, \alpha) &= \lambda\left(s|p^+\right) \geq \lambda(s|y) \text{ for } y \in h_f(s|p) \end{aligned}$$

□

Lemma 19. *If* $p \in P_f$ *then* $\lambda_0 (p(s), -\gamma, \alpha) = \min_{p \in h_f(s|p)} \lambda (s|p)$

Proof. If $p \in P_f$ then let $p^- \in h_f(s|p)$ such that

$$p^-(t) = \begin{cases} p(t) & \text{if } t < s \\ f\left(f^{-1}(p(s)) - \gamma[t-s]\right) & \text{if } t \geq s \end{cases}$$

Hence for all $y \in h_f(s|p)$

$$\begin{aligned} p^-(t) &\leq y(t) \\ E_s\left\{p^-(T)\right\} &\leq E_s\{y(T)\} \\ \alpha\bar{p} - p(s) + (1-\alpha) E_s\left\{p^-(T)\right\} &\leq \alpha\bar{p} - p(s) + (1-\alpha) E_s\{y(T)\} \end{aligned}$$

So $\lambda(s|p^-) \leq \lambda(s|y)$ by Lemma 6. By the definition of g_f we have

$$\begin{aligned} g_f(p(s), -\gamma, t-s) = p^-(t) \quad \text{for } t \geq s \\ E\{g_f(p(s), -\gamma, T)\} = E_s\left\{p^-(T)\right\} \\ \lambda(p(s), -\gamma, \alpha) = \lambda\left(s|p^-\right) \leq \lambda(s|y) \text{ for } y \in h_f(s|p) \end{aligned}$$

□

Lemma 20. *If* $p(t) \in [\check{p}_0(\alpha), \hat{p}_0(\alpha)]$ *then* $\lambda(t|p^\circ) = 0$ *for some* $p^\circ \in h_f(t|p)$

Proof. If $p(t) \in [\check{p}_0(\alpha), \hat{p}_0(\alpha)]$ then by definition of $\check{p}_0$ and $\hat{p}_0$

$$\lambda_0(p(t), -\gamma, \alpha) \le 0 \le \lambda_0(p(t), \gamma, \alpha)$$

Since $\lambda_0(x, c, a)$ is continuous in c there exists $c_0 \in [-\gamma, \gamma]$ such that

$$\lambda_0(p(t), c_0, \alpha) = 0$$

Now let $p^\circ \in h_f(t|p)$ such that

$$p^\circ(\tau) = \begin{cases} p(\tau) & \text{if } \tau \le t \\ g_f(p(t), c_0, \tau - t) & \text{if } \tau > t \end{cases}$$

Hence by Lemma 6 we have

$$\begin{aligned} \lambda(t|p^\circ) &= \alpha\bar{p} - p(t) + (1-\alpha) E_t\{p^\circ(T)\} \\ &= \alpha\bar{p} - p(t) + (1-\alpha) E\{g_f(p(t), c_0, T)\} \\ &= \lambda_0(p(t), c_0, \alpha) = 0 \end{aligned}$$

□

Lemma 21. $h_0^{\lhd}(\alpha) \subseteq h_0(t^*|p^*)$

Proof. If $p \in h_0^{\lhd}(\alpha)$ then $p \in p_f(t^*|p^*)$ and $p(t) \in [\check{p}_0(\alpha), \hat{p}_0(\alpha)]$ for all $t \ge t^*$. Since $p \in p_f(t^*|p^*)$ then $p = p_\phi$ for some $\phi \in \Phi$. By Lemma 20 for all $t \ge t^*$ there exists $p_t^\circ \in h_f(t|p)$ such that $\lambda(t|p_t^\circ) = 0$. Then let $\mu_i \in \mathcal{M}_i^0$ such that

$$\mu_i(p_t^\circ | h(t|p)) = 1 \quad \text{for } t \ge t^*$$

Then by definition of $\lambda(h|\mu_i)$ we have

$$\begin{aligned} \lambda(h(t|p)|\mu_i) &= \int_P \lambda(t|y)\, d\mu_i(y|h(t|p)) \\ &= \lambda(t|p_s^\circ) = 0 \end{aligned}$$

Hence $\phi \in \Phi^*(\mu_i)$ by Lemma 16 so $p = p_\phi \in P_0 \cap h_f(t^*|p^*) = h_0(t^*|p^*)$. □

Lemma 22. $h_0\left(t^*|p^*\right) \subseteq h_0^{\triangleleft}\left(\alpha\right)$

Proof. If $p \in p_0\left(t^*|p^*\right)$ then there exists $\mu \in \mathcal{M}^0$ and $\phi \in \Phi^*\left(\mu\right)$ such that $p = p_\phi$. Then by Lemma 16 we can write

$$\phi_i\left(h(t|p)\right) \in \underset{x\in[-\gamma,\gamma]}{\operatorname{argmax}}\, x\,\lambda\left(h(t|p)|\mu_i\right)$$

Since $\lambda_0\left(x,\gamma,\alpha\right)$ is continuous in x we have

$$\hat{p}_0\left(\alpha\right) = \inf\left\{x > \bar{p} : \lambda_0\left(x,\gamma,\alpha\right) < 0\right\}$$
$$\exists \hat{y} > \hat{p}_0\left(\alpha\right) \quad \forall x \in \left(\hat{p}_0\left(\alpha\right), \hat{y}\right),\ \lambda_0\left(x,\gamma,\alpha\right) < 0$$

By Lemma 18 we have $\lambda_0\left(p\left(t\right),\gamma,\alpha\right) = \max\limits_{\mu_i\in\mathcal{M}_i^0} \lambda\left(h(t|p)|\mu_i\right)$ so

$$\begin{aligned}
p\left(t\right) \in \left(\hat{p}_0\left(\alpha\right), \hat{y}\right) &\implies \lambda_0\left(p\left(t\right),\gamma,\alpha\right) < 0 \\
&\implies \lambda\left(h(t|p)|\mu_i\right) < 0 \quad \text{for } i \in [0,1] \\
&\implies \phi_i\left(h(t|p)\right) = -\gamma \quad \text{for } i \in [0,1] \\
&\implies \dot{p}\left(t\right) < 0
\end{aligned}$$

Since $\lambda_0\left(x,-\gamma,\alpha\right)$ is continuous in x we have

$$\check{p}_0\left(\alpha\right) = \sup\left\{x < \bar{p} : \lambda_0^-\left(x,-\gamma,\alpha\right) > 0\right\}$$
$$\exists \check{y} < \hat{p}_0\left(\alpha\right) \quad \forall x \in \left(\check{y}, \check{p}_0\left(\alpha\right)\right),\ \lambda_0\left(x,-\gamma,\alpha\right) > 0$$

By Lemma 19 we have $\lambda_0\left(p\left(t\right),-\gamma,\alpha\right) = \min\limits_{\mu_i\in\mathcal{M}_i^0} \lambda\left(h(t|p)|\mu_i\right)$ so

$$\begin{aligned}
p\left(t\right) \in \left(\check{y}, \check{p}_0\left(\alpha\right)\right) &\implies \lambda_0\left(p\left(t\right),-\gamma,\alpha\right) > 0 \\
&\implies \lambda\left(h(t|p)|\mu_i\right) > 0 \quad \text{for } i \in [0,1] \\
&\implies \phi_i\left(h(t|p)\right) = \gamma \quad \text{for } i \in [0,1] \\
&\implies \dot{p}\left(t\right) > 0
\end{aligned}$$

Since $p \in p_0\left(t^*|p^*\right)$ we have $p\left(t^*\right) = \bar{p} \in \left[\check{p}_0\left(\alpha\right), \hat{p}_0\left(\alpha\right)\right]$ so

$$p\left(t\right) \in \left[\check{p}_0\left(\alpha\right), \hat{p}_0\left(\alpha\right)\right] \text{ for all } t > t^*$$

Hence $p \in h_0^{\triangleleft}\left(\alpha\right) = \left\{p \in p_f\left(t^*|p^*\right) : p\left(t\right) \in \left[\check{p}_0\left(\alpha\right), \hat{p}_0\left(\alpha\right)\right] \text{ for } t \geq t^*\right\}$. □

Lemma 23. $h_0\left(t^*|p^*\right) = h_0^{\triangleleft}\left(\alpha\right)$

Proof. By Lemma 21 we have $h_0^{\triangleleft}\left(\alpha\right) \subseteq h_0\left(t^*|p^*\right)$ and by Lemma 22 we have $h_0\left(t^*|p^*\right) \subseteq h_0^{\triangleleft}\left(\alpha\right)$. Thus $h_0\left(t^*|p^*\right) = h_0^{\triangleleft}\left(\alpha\right)$. □

Lemma 24. $\hat{p}_0\left(\alpha\right)$ *is decreasing in* α

Proof. If $x \geq \bar{p}$ then by the definition of g_f

$$\begin{aligned} g_f\left(x,\gamma,t\right) &= f\left(f^{-1}\left(x\right)+\gamma t\right) \\ E\left\{g_f\left(x,\gamma,T\right)\right\} &= \int_0^{\infty} e^{-\beta t} f\left(f^{-1}\left(x\right)+\gamma t\right) dt \\ E\left\{g_f\left(x,\gamma,T\right)\right\} &> x \geq \bar{p} \end{aligned}$$

By the definition of λ_0 we can write

$$\begin{aligned} \lambda_0\left(x,\gamma,\alpha\right) &= \alpha\bar{p} - x + \left(1-\alpha\right) E\left\{g_f\left(x,\gamma,T\right)\right\} \\ \frac{\partial\lambda_0\left(x,\gamma,\alpha\right)}{\partial\alpha} &= \bar{p} - E\left\{g_f\left(x,\gamma,T\right)\right\} \\ \frac{\partial\lambda_0\left(x,\gamma,\alpha\right)}{\partial\alpha} &< 0 \quad \text{ since } E\left\{g_f\left(x,\gamma,T\right)\right\} > \bar{p} \end{aligned}$$

Hence if $\alpha' \in \left(\alpha,1\right)$ then

$$\begin{aligned} \lambda_0\left(x,\gamma,\alpha'\right) < \lambda_0\left(x,\gamma,\alpha\right) \quad \text{ for all } x \geq \bar{p} \\ \inf\left\{x > \bar{p} : \lambda_0\left(x,\gamma,\alpha'\right) < 0\right\} \leq \inf\left\{x > \bar{p} : \lambda_0\left(x,\gamma,\alpha\right) < 0\right\} \end{aligned}$$

So by definition of $\hat{p}_0$ we have $\hat{p}_0\left(\alpha'\right) \leq \hat{p}_0\left(\alpha\right)$. □

Lemma 25. *$\check{p}_0(\alpha)$ is increasing in α*

Proof. If $x \leq \bar{p}$ then by the definition of g_f

$$\begin{aligned} g_f(x,-\gamma,t) &= f\left(f^{-1}(x)-\gamma t\right) \\ E\{g_f(x,-\gamma,T)\} &= \int_0^\infty e^{-\beta t} f\left(f^{-1}(x)-\gamma t\right) dt \\ E\{g_f(x,-\gamma,T)\} &< x \leq \bar{p} \end{aligned}$$

By the definition of λ_0 we can write

$$\begin{aligned} \lambda_0(x,-\gamma,\alpha) &= \alpha\bar{p} - x + (1-\alpha) E\{g_f(x,-\gamma,T)\} \\ \frac{\partial\lambda_0(x,-\gamma,\alpha)}{\partial\alpha} &= \bar{p} - E\{g_f(x,-\gamma,T)\} \\ \frac{\partial\lambda_0(x,-\gamma,\alpha)}{\partial\alpha} &> 0 \quad \text{since } E\{g_f(x,-\gamma,T)\} < \bar{p} \end{aligned}$$

Hence if $\alpha' \in (\alpha, 1)$ then

$$\begin{aligned} \lambda_0(x,-\gamma,\alpha') &> \lambda_0(x,-\gamma,\alpha) \quad \text{for all } x \leq \bar{p} \\ \sup\{x < \bar{p} : \lambda_0(x,-\gamma,\alpha') > 0\} &\geq \sup\{x < \bar{p} : \lambda_0(x,-\gamma,\alpha) > 0\} \end{aligned}$$

So by definition of $\check{p}_0$ we have $\check{p}_0(\alpha') \geq \check{p}_0(\alpha)$. $\square$

Lemma 26. $\lim_{\alpha \to 1} \hat{p}_0(\alpha) = \bar{p}$

Proof. By the definition of $\hat{p}_0$ we can write

$$\begin{aligned} \hat{p}_0(\alpha) &= \inf\{x > \bar{p} : \lambda_0(x, \gamma, \alpha) < 0\} \\ \hat{p}_0(\alpha) &\geq \bar{p} \end{aligned}$$

By the definitions of λ_0 and g_f we have

$$\begin{aligned} \lambda_0(x, \gamma, \alpha) &= \alpha\bar{p} - x + (1-\alpha) E\{g_f(x, \gamma, T)\} \\ \lim_{\alpha \to 1} \lambda_0(x, \gamma, \alpha) &= \bar{p} - x \end{aligned}$$

Hence for all $x > \bar{p}$ there exists $A(x) \in (0, 1)$ such that

$$\begin{aligned} \alpha \in (A(x), 1) &\implies \lambda_0(x, \gamma, \alpha) < 0 \\ &\implies \hat{p}_0(\alpha) \in [\bar{p}, x] \end{aligned}$$

Thus $\lim_{\alpha \to 1} \hat{p}_0(\alpha) = \bar{p}$ by the squeeze theorem. □

Lemma 27. $\lim_{\alpha \to 1} \check{p}_0(\alpha) = \bar{p}$

Proof. By the definition of $\check{p}_0$ we can write

$$\begin{aligned} \check{p}_0(\alpha) &= \sup\{x < \bar{p} : \lambda_0(x, -\gamma, \alpha) > 0\} \\ \check{p}_0(\alpha) &\leq \bar{p} \end{aligned}$$

By the definitions of λ_0 and g_f we have

$$\begin{aligned} \lambda_0(x, -\gamma, \alpha) &= \alpha\bar{p} - x + (1-\alpha) E\{g_f(x, -\gamma, T)\} \\ \lim_{\alpha \to 1} \lambda_0(x, -\gamma, \alpha) &= \bar{p} - x \end{aligned}$$

Hence for all $x < \bar{p}$ there exists $A(x) \in (0, 1)$ such that

$$\begin{aligned} \alpha \in (A(x), 1) &\implies \lambda_0(x, -\gamma, \alpha) > 0 \\ &\implies \check{p}_0(\alpha) \in [x, \bar{p}] \end{aligned}$$

Thus $\lim_{\alpha \to 1} \check{p}_0(\alpha) = \bar{p}$ by the squeeze theorem. □

Lemma 28. $\lim_{\alpha \to 0} \hat{p}_0(\alpha) = \infty$

Proof. If $x \geq \bar{p}$ then by the definition of g_f we can write

$$\begin{aligned} g_f(x, \gamma, t) &= f\left(f^{-1}(x) + \gamma t\right) \\ E\{g_f(x, \gamma, T)\} &= \int_0^\infty e^{-\beta t} f\left(f^{-1}(x) + \gamma t\right) dt \\ E\{g_f(x, \gamma, T)\} &> x \geq \bar{p} \end{aligned}$$

Since $g_f(x, c, t)$ is continuous in x, it attains a minimum on every compact interval. So if $y > \bar{p}$ then there exists $A(y), B(y)$ such that

$$\begin{aligned} 0 > A(y) &= \min_{x \in [\bar{p}, y]} E\{g_f(x, \gamma, T)\} - x \\ 0 < B(y) &= \min_{x \in [\bar{p}, y]} \bar{p} - E\{g_f(x, \gamma, T)\} \end{aligned}$$

By the definition of λ_0, for all $x \in [\bar{p}, y]$ we have

$$\begin{aligned} \lambda_0(x, \gamma, \alpha) &= \alpha \bar{p} - x + (1 - \alpha) E\{g_f(x, \gamma, T)\} \\ \lambda_0(x, \gamma, \alpha) &= E\{g_f(x, \gamma, T)\} - x + \alpha [\bar{p} - E\{g_f(x, \gamma, T)\}] \\ \lambda_0(x, \gamma, \alpha) &\geq A(y) + \alpha B(y) \end{aligned}$$

Since $A(y) > 0$ then for all $y > \bar{p}$ there exists $K(y) > 0$ such that

$$\alpha \in (0, K(y)) \implies \lambda_0(x, \gamma, \alpha) \geq 0 \quad \text{for all } x \in [\bar{p}, y]$$

Hence for all $y > \bar{p}$ we have

$$\alpha \in (0, K(y)) \implies \hat{p}_0(\alpha) = \inf\{x > \bar{p} : \lambda_0(x, \gamma, \alpha) < 0\} \geq y$$

Therefore $\lim_{\alpha \to 0} \hat{p}_0(\alpha) = \infty$. □

Lemma 29. $\lim_{\alpha \to 0} \check{p}_0(\alpha) = 0$

Proof. If $x \leq \bar{p}$ then by the definition of g_f we can write

$$\begin{aligned} g_f(x, -\gamma, t) &= f\left(f^{-1}(x) - \gamma t\right) \\ E\{g_f(x, -\gamma, T)\} &= \int_0^\infty e^{-\beta t} f\left(f^{-1}(x) - \gamma t\right) dt \\ E\{g_f(x, -\gamma, T)\} &< x \leq \bar{p} \end{aligned}$$

Since $g_f(x, c, t)$ is continuous in x, it attains a maximum on every compact interval. So if $0 < y < \bar{p}$ then there exists $A(y), B(y)$ such that

$$\begin{aligned} 0 < A(y) &= \max_{x \in [y, \bar{p}]} E\{g_f(x, -\gamma, T)\} - x \\ 0 > B(y) &= \max_{x \in [y, \bar{p}]} \bar{p} - E\{g_f(x, -\gamma, T)\} \end{aligned}$$

By the definition of λ_0, for all $x \in [y, \bar{p}]$ we have

$$\begin{aligned} \lambda_0(x, -\gamma, \alpha) &= \alpha\bar{p} - x + (1 - \alpha) E\{g_f(x, -\gamma, T)\} \\ \lambda_0(x, -\gamma, \alpha) &= E\{g_f(x, -\gamma, T)\} - x + \alpha\left[\bar{p} - E\{g_f(x, -\gamma, T)\}\right] \\ \lambda_0(x, -\gamma, \alpha) &\leq A(y) + \alpha B(y) \end{aligned}$$

Since $A(y) < 0$ then for all $0 < y < \bar{p}$ there exists $K(y) > 0$ such that

$$\alpha \in (0, K(y)) \implies \lambda_0(x, -\gamma, \alpha) \leq 0 \quad \text{for all } x \in [y, \bar{p}]$$

Hence for all $0 < y < \bar{p}$ we have

$$\alpha \in (0, K(y)) \implies 0 < \check{p}_0(\alpha) = \sup\{x < \bar{p} : \lambda_0(x, -\gamma, \alpha) > 0\} \leq y$$

Therefore $\lim_{\alpha \to 0} \check{p}_0(\alpha) = 0$. □

A.5 Knowledge of Rationality

Definition 7. For $n \in \mathbb{N}_0$, $a \in (0,1)$, $x \in [\hat{p}_n(a), \check{p}_n(a)]$, $c \in [-\gamma, \gamma]$, $t \in \mathbb{R}_+$

$$t_n(x,c,a) = \begin{cases} c^{-1}\left|f^{-1}(\hat{p}_n(a)) - f^{-1}(x)\right| & \text{if } c > 0 \\ 0 & \text{if } c = 0 \\ c^{-1}\left|f^{-1}(\check{p}_n(a)) - f^{-1}(x)\right| & \text{if } c < 0 \end{cases}$$

$$g_n(x,c,a,t) = \begin{cases} f\left(f^{-1}(x) + ct\right) & \text{if } t < t_n(x,a,c) \text{ or } c = 0 \\ \hat{p}_n(a) & \text{if } t \geq t_n(x,a,c) \text{ and } c > 0 \\ \check{p}_n(a) & \text{if } t \geq t_n(x,a,c) \text{ and } c < 0 \end{cases}$$

$$\lambda_{n+1}(x,c,a) = a\bar{p} - x + (1-a)\,E\left\{g_n(x,c,a,T)\right\}$$

$$\hat{p}_{n+1}(a) = \inf\left\{x \in [\bar{p}, \hat{p}_n(a)] : \lambda_{n+1}(x,\gamma,a) < 0\right\}$$

$$\check{p}_{n+1}(a) = \sup\left\{x \in [\check{p}_n(a), \bar{p}] : \lambda_{n+1}(x,-\gamma,a) > 0\right\}$$

$$h_n^{\triangleleft}(a) = \left\{p \in p_f(t^*|p^*) : p(t) \in [\check{p}_n(a), \hat{p}_n(a)] \text{ for } t \geq t^*\right\}$$

Lemma 30. $\check{p}_n(a) < \bar{p} < \hat{p}_n(a)$ *for all* $n \in \mathbb{N}$ *and* $a \in (0,1)$

Proof. By Lemma 17 we have

$$\check{p}_0(a) < \bar{p} < \hat{p}_0(a) \quad \text{for } a \in (0,1)$$

For the inductive hypothesis, suppose that

$$\check{p}_n(a) < \bar{p} < \hat{p}_n(a)$$

By the definition of g_n we can write

$$E\{g_n(x,\gamma,a,T)\} = \beta \int_0^{t_n(x,\gamma,a)} e^{-\beta t} f\left(f^{-1}(x) + \gamma t\right) dt + \hat{p}_n(a) e^{-\beta t_n(x,\gamma,a)}$$
$$E\{g_n(x,-\gamma,a,T)\} = \beta \int_0^{t_n(x,-\gamma,a)} e^{-\beta t} f\left(f^{-1}(x) - \gamma t\right) dt + \check{p}_n(a) e^{-\beta t_n(x,-\gamma,a)}$$

So if $x \in [\check{p}_n(a), \hat{p}_n(a)]$ then

$$E\{g_n(x,-\gamma,T)\} < x < E\{g_n(x,\gamma,T)\}$$

By the definition of λ_n we can write

$$\begin{aligned}
\lambda_{n+1}(\bar{p},c,a) &= a\bar{p} - \bar{p} + (1-a) E\{g_n(\bar{p},c,a,T)\} \\
\lambda_{n+1}(\bar{p},c,a) &= (1-a) E\{g_n(\bar{p},c,a,T) - \bar{p}\} \\
\lambda_{n+1}(\bar{p},\gamma,a) &> 0 \quad \text{since } E\{g_n(\bar{p},\gamma,T)\} > \bar{p} \\
\lambda_{n+1}(\bar{p},-\gamma,a) &< 0 \quad \text{since } E\{g_n(\bar{p},-\gamma,T)\} < \bar{p}
\end{aligned}$$

Since $\lambda_{n+1}(x,c,a)$ is continuous in x there exists $k > 0$ such that

$$\lambda_{n+1}(\bar{p}-\varepsilon,-\gamma,a) < 0 < \lambda_{n+1}(\bar{p}+\varepsilon,\gamma,a) \qquad \text{for all } \varepsilon \in (0,k)$$

So by the definition of $\hat{p}_{n+1}$ and $\check{p}_{n+1}$

$$\begin{aligned}
\hat{p}_{n+1}(a) &= \inf\{x \in [\bar{p}, \hat{p}_n(a)] : \lambda_{n+1}(x,\gamma,a) < 0\} > \bar{p} \\
\check{p}_{n+1}(a) &= \sup\{x \in [\check{p}_n(a), \bar{p}] : \lambda_{n+1}(x,-\gamma,a) > 0\} < \bar{p}
\end{aligned}$$

□

Lemma 31. *for all* $a \in (0,1)$ *and* $n \in \mathbb{N}_0$

$$\check{p}_{n+1}(a) < \check{p}_n(a) \leq \hat{p}_n(a) < \hat{p}_{n+1}(a)$$

Proof. If $p \in h_n^{\triangleleft}(a)$ then $p \in p_f(t^*|p^*)$ and

$$p(t) \in [\check{p}_n(a), \hat{p}_n(a)] \text{ for } t \geq t^*$$

By the definition of g_n we have

$$\begin{aligned} g_n(\hat{p}_n(a), \gamma, a, t) &= \hat{p}_n(a) \qquad \text{for all } t \in \mathbb{R}_+ \\ g_n(\check{p}_n(a), -\gamma, a, t) &= \check{p}_n(a) \qquad \text{for all } t \in \mathbb{R}_+ \end{aligned}$$

Hence by Lemma 30 and the definition of λ_{n+1} we have

$$\begin{aligned} \lambda_{n+1}(\hat{p}_n(a), \gamma, a) &= a\bar{p} - \hat{p}_n(a) + (1-a) E\{g_n(\hat{p}_n(a), \gamma, a, T)\} \\ \lambda_{n+1}(\hat{p}_n(a), \gamma, a) &= a\bar{p} - \hat{p}_n(a) + (1-a)\hat{p}_n(a) \\ \lambda_{n+1}(\hat{p}_n(a), \gamma, a) &= a[\bar{p} - \hat{p}_n(a)] < 0 \\ \lambda_{n+1}(\check{p}_n(a), -\gamma, a) &= a\bar{p} - \check{p}_n(a) + (1-a) E\{g_n(\check{p}_n(a), -\gamma, a, T)\} \\ \lambda_{n+1}(\check{p}_n(a), -\gamma, a) &= a\bar{p} - \check{p}_n(a) + (1-a)\check{p}_n(a) \\ \lambda_{n+1}(\check{p}_n(a), -\gamma, a) &= a[\bar{p} - \check{p}_n(a)] > 0 \end{aligned}$$

Since $\lambda_{n+1}(x, c, a)$ is continuous in x there exists $\varepsilon > 0$ such that

$$\begin{aligned} \lambda_{n+1}(\hat{p}_n(a) - \varepsilon, \gamma, a) &< 0 \\ \lambda_{n+1}(\check{p}_n(a) + \varepsilon, -\gamma, a) &> 0 \end{aligned}$$

Thus by the definition of $\hat{p}_{n+1}$ and $\check{p}_{n+1}$

$$\begin{aligned} \hat{p}_{n+1}(a) &= \inf\{x \in [\bar{p}, \hat{p}_n(a)] : \lambda_{n+1}(x, \gamma, a) < 0\} < \hat{p}_n(a) \\ \check{p}_{n+1}(a) &= \sup\{x \in [\check{p}_n(a), \bar{p}] : \lambda_{n+1}(x, -\gamma, a) > 0\} > \check{p}_n(a) \end{aligned}$$

□

Lemma 32. *If* $p \in h_n(t^*|p^*) = h_n^{\lhd}(\alpha)$ *and* $s \geq t^*$ *then*

$$\lambda_{n+1}(p(s), \gamma, \alpha) = \max_{p \in h_n(s|p)} \lambda(s|p)$$

Proof. If $p \in h_n(t^*|p^*) = h_n^{\lhd}(\alpha)$ and $s \geq t^*$ then let $p^+ \in h_n(s|p)$ such that

$$p^+(t) = \begin{cases} p(t) & \text{if } t < s \\ g_n(p(s), \gamma, \alpha, t-s) & \text{if } t \geq s \end{cases}$$

Hence if $y \in h_n(s|p)$ then by the definition of g_n

$$\begin{aligned} p^+(t) &\geq y(t) \\ E_s\left\{p^+(T)\right\} &\geq E_s\{y(T)\} \\ \alpha\bar{p} - p(s) + (1-\alpha) E_s\left\{p^+(T)\right\} &\geq \alpha\bar{p} - p(s) + (1-\alpha) E_s\{y(T)\} \end{aligned}$$

Then by Lemma 6 we can write

$$\lambda\left(s|p^+\right) \geq \lambda(s|y) \qquad \text{for all } y \in h_n(s|p)$$

And by the definition of λ_{n+1} we have

$$\begin{aligned} g_n(p(s), \gamma, \alpha, t-s) &= p^+(t) \quad \text{for } t \geq s \\ E\{g_n(p(s), \gamma, \alpha, T)\} &= E_s\left\{p^+(T)\right\} \\ \lambda_{n+1}(p(s), \gamma, \alpha) &= \lambda\left(s|p^+\right) \geq \lambda(s|y) \quad \text{for } y \in h_n(s|p) \end{aligned}$$

□

Lemma 33. *If* $p \in h_n(t^*|p^*) = h_n^{\triangleleft}(\alpha)$ *and* $s \geq t^*$ *then*

$$\lambda_{n+1}(p(s), -\gamma, \alpha) = \min_{p \in h_n(s|p)} \lambda(s|p)$$

Proof. If $p \in h_n(t^*|p^*) = h_n^{\triangleleft}(\alpha)$ and $s \geq t^*$ then let $p^- \in h_n(s|p)$ such that

$$p^-(t) = \begin{cases} p(t) & \text{if } t < s \\ g_n(p(s), -\gamma, \alpha, t-s) & \text{if } t \geq s \end{cases}$$

Hence if $y \in h_n(s|p)$ then by the definition of g_n

$$\begin{aligned} p^-(t) &\leq y(t) \\ E_s\left\{p^-(T)\right\} &\leq E_s\{y(T)\} \\ \alpha\bar{p} - p(s) + (1-\alpha)E_s\left\{p^-(T)\right\} &\leq \alpha\bar{p} - p(s) + (1-\alpha)E_s\{y(T)\} \end{aligned}$$

Then by Lemma 6 we can write

$$\lambda\left(s|p^-\right) \leq \lambda(s|y) \quad \text{for all } y \in h_n(s|p)$$

And by the definition of λ_{n+1} we have

$$\begin{aligned} g_n(p(s), -\gamma, \alpha, t-s) &= p^-(t) \quad \text{for } t \geq s \\ E\{g_n(p(s), -\gamma, \alpha, T)\} &= E_s\left\{p^-(T)\right\} \\ \lambda_{n+1}(p(s), -\gamma, \alpha) &= \lambda\left(s|p^-\right) \leq \lambda(s|y) \quad \text{for } y \in h_n(s|p) \end{aligned}$$

□

Lemma 34. *If* $p \in h_n(t^*|p^*) = h_n^{\triangleleft}(\alpha)$ *and* $t \geq t^*$ *then*

$$\lambda(t|p^\circ) = 0 \quad \text{for some } p^\circ \in h_n(t|p)$$

Proof. If $p \in h_n(t^*|p^*) = h_n^{\triangleleft}(\alpha)$ and $t \geq t^*$ then by the definition of $h_n^{\triangleleft}$

$$p(t) \in [\check{p}_n(\alpha), \hat{p}_n(\alpha)]$$

Hence by the definitions of $\check{p}_n$ and $\hat{p}_n$ and Lemma 30

$$\lambda_n(p(t), -\gamma, \alpha) \leq 0 \leq \lambda_n(p(t), \gamma, \alpha)$$

Since $\lambda_n(x, c, a)$ is continuous in c there exists $c_0 \in [-\gamma, \gamma]$ such that

$$\lambda_n(p(t), c_0, \alpha) = 0$$

Since $h_n(t^*|p^*) = h_n^{\triangleleft}(\alpha)$ there exists $p^\circ \in h_n(t|p)$ such that

$$p^\circ(\tau) = \begin{cases} p(\tau) & \text{if } \tau \leq t \\ g_n(p(t), c_0, \alpha, \tau - t) & \text{if } \tau > t \end{cases}$$

Hence by Lemma 6 we have

$$\begin{aligned} \lambda(t|p^\circ) &= \alpha\bar{p} - p(t) + (1-\alpha) E_t\{p^\circ(T)\} \\ &= \alpha\bar{p} - p(t) + (1-\alpha) E\{g_n(p(t), c_0, \alpha, T)\} \\ &= \lambda_n(p(t), c_0, \alpha) = 0 \end{aligned}$$

□

Lemma 35. *If* $h_{n-1}(t^*|p^*) = h^{\triangleleft}_{n-1}(\alpha)$ *then* $h^{\triangleleft}_{n}(\alpha) \subseteq h_n(t^*|p^*)$

Proof. If $p \in h^{\triangleleft}_{n}(\alpha)$ then $p \in p_f(t^*|p^*)$ and $p(t) \in [\check{p}_n(\alpha), \hat{p}_n(\alpha)]$ for all $t \geq t^*$. Since $p \in p_f(t^*|p^*)$ then $p = p_\phi$ for some $\phi \in \Phi$. Then $h^{\triangleleft}_{n}(\alpha) \subseteq h^{\triangleleft}_{n-1}(\alpha)$ by Lemma 31 so $p \in h^{\triangleleft}_{n-1}(\alpha)$. If $h_{n-1}(t^*|p^*) = h^{\triangleleft}_{n-1}(\alpha)$ then by Lemma 20 for all $t \geq t^*$ there exists $p^\circ_t \in h_{n-1}(t|p)$ such that $\lambda(t|p^\circ_t) = 0$. Then let $\mu_i \in \mathcal{M}^n_i$ such that

$$\mu_i(p^\circ_t|h(t|p)) = 1 \quad \text{for } t \geq t^*$$

Then by definition of $\lambda(h|\mu_i)$ we have

$$\begin{aligned}\lambda(h(t|p)|\mu_i) &= \int_P \lambda(t|y)\, d\mu_i(y|h(t|p)) \\ &= \lambda(t|p^\circ_s) = 0\end{aligned}$$

Hence $\phi \in \Phi^*(\mu_i)$ by Lemma 16 so $p = p_\phi \in P_n \cap h_f(t^*|p^*) = h_n(t^*|p^*)$. □

Lemma 36. *If* $h_{n-1}(t^*|p^*) = h^{\triangleleft}_{n-1}(\alpha)$ *then* $h_n(t^*|p^*) \subseteq h^{\triangleleft}_n(\alpha)$

Proof. If $p \in p_n(t^*|p^*)$ then there exists $\mu \in \mathcal{M}^n$ and $\phi \in \Phi^*(\mu)$ such that $p = p_\phi$. Then by Lemma 16 we can write

$$\phi_i(h(t|p)) \in \underset{x\in[-\gamma,\gamma]}{\operatorname{argmax}}\, x\,\lambda(h(t|p)|\mu_i)$$

by Lemma 31, since $\lambda_n(x,\gamma,\alpha)$ is continuous in x we have

$$\begin{aligned}
\hat{p}_n(\alpha) = \inf\{x \in [\bar{p}, \hat{p}_{n-1}(\alpha)] : \lambda_n(x,\gamma,\alpha) < 0\} < \hat{p}_{n-1}(\alpha) \\
\check{p}_n(\alpha) = \sup\{x \in [\check{p}_{n-1}(\alpha), \bar{p}] : \lambda_n(x,-\gamma,\alpha) > 0\} > \check{p}_{n-1}(\alpha) \\
\exists \hat{y} \in (\hat{p}_n(\alpha), \hat{p}_{n-1}(\alpha)) \quad \forall x \in (\hat{p}_n(\alpha), \hat{y}),\ \lambda_n(x,\gamma,\alpha) < 0 \\
\exists \check{y} \in (\check{p}_{n-1}(\alpha), \hat{p}_n(\alpha)) \quad \forall x \in (\check{y}, \check{p}_n(\alpha)),\ \lambda_n(x,\gamma,\alpha) > 0
\end{aligned}$$

If $h_{n-1}(t^*|p^*) = h^{\triangleleft}_{n-1}(\alpha)$ then by Lemma 32 we have

$$\begin{aligned}
\lambda_n(p(t),\gamma,\alpha) &= \max_{\mu_i\in\mathcal{M}^{n-1}_i} \lambda(h(t|p)|\mu_i) \\
p(t) \in (\hat{p}_n(\alpha), \hat{y}) &\implies \lambda_n(p(t),\gamma,\alpha) < 0 \\
&\implies \lambda(h(t|p)|\mu_i) < 0 \quad \text{for } i \in [0,1] \\
&\implies \phi_i(h(t|p)) = -\gamma \quad \text{for } i \in [0,1] \\
&\implies \dot{p}(t) < 0
\end{aligned}$$

And by Lemma 33 we have

$$\begin{aligned}
\lambda_n(p(t),-\gamma,\alpha) &= \min_{\mu_i\in\mathcal{M}^{n-1}_i} \lambda(h(t|p)|\mu_i) \\
p(t) \in (\check{y}, \check{p}_n(\alpha)) &\implies \lambda_n(p(t),-\gamma,\alpha) > 0 \\
&\implies \lambda(h(t|p)|\mu_i) > 0 \quad \text{for } i \in [0,1] \\
&\implies \phi_i(h(t|p)) = \gamma \quad \text{for } i \in [0,1] \\
&\implies \dot{p}(t) > 0
\end{aligned}$$

Since $p \in p_n(t^*|p^*)$ we have $p(t^*) = \bar{p} \in [\check{p}_n(\alpha), \hat{p}_n(\alpha)]$ so

$$p(t) \in [\check{p}_n(\alpha), \hat{p}_n(\alpha)] \text{ for all } t > t^*$$

Hence $p \in h^{\triangleleft}_n(\alpha) = \{p \in p_f(t^*|p^*) : p(t) \in [\check{p}_n(\alpha), \hat{p}_n(\alpha)] \text{ for } t \geq t^*\}$. □

Lemma 37. $h_n(t^*|p^*) = h_n^{\triangleleft}(\alpha)$ *for all* $n \in \mathbb{N}$

Proof. By Lemma 23 we have $h_0(t^*|p^*) = h_0^{\triangleleft}(\alpha)$. Suppose that $h_{n-1}(t^*|p^*) = h_{n-1}^{\triangleleft}(\alpha)$. Then $h_n^{\triangleleft}(\alpha) \subseteq h_n(t^*|p^*)$ by Lemma 21 and $h_n(t^*|p^*) \subseteq h_n^{\triangleleft}(\alpha)$ by Lemma 22. Hence $h_n(t^*|p^*) = h_n^{\triangleleft}(\alpha)$. □

Lemma 38. $\dfrac{dE\{g_n(x,\gamma,\alpha,T)\}}{d\hat{p}_n(\alpha)} = e^{-\beta t_n(x,\alpha,\gamma)}$

Proof. By the definition of t_n we have

$$t_n(x,\gamma,a) = c^{-1}\left|f^{-1}(\hat{p}_n(a)) - f^{-1}(x)\right|$$

Differentiating with respect to $\hat{p}_n(a)$ obtains

$$\frac{\partial t_n(x,\gamma,a)}{\partial \hat{p}_n(a)} = \frac{\operatorname{sgn}(\hat{p}_n(a) - x)}{cf'(f^{-1}(\hat{p}_n(a)))}$$

By the definition of g_n we have

$$E\{g_n(x,\gamma,\alpha,T)\} = \beta\int_0^{t_n(x,\alpha,\gamma)} e^{-\beta t} f\left(f^{-1}(x) + \gamma t\right) dt + e^{-\beta t_n(x,\alpha,\gamma)}\hat{p}_n(\alpha)$$

Differentiating with respect to $t_n(x,\alpha,\gamma)$ obtains

$$\frac{\partial E\{g_n(x,\gamma,\alpha,T)\}}{\partial t_n(x,\gamma,a)} = \beta e^{-\beta t_n(x,\alpha,\gamma)}\hat{p}_n(\alpha) - \beta e^{-\beta t_n(x,\alpha,\gamma)}\hat{p}_n(\alpha) = 0$$

Differentiating with respect to $\hat{p}_n(\alpha)$ obtains

$$\begin{aligned}\frac{dE\{g_n(x,\gamma,\alpha,T)\}}{d\hat{p}_n(\alpha)} &= \frac{\partial E\{g_n(x,\gamma,\alpha,T)\}}{\partial t_n(x,\gamma,a)}\frac{\partial t_n(x,\gamma,a)}{\partial \hat{p}_n(a)} + \frac{\partial E\{g_n(x,\gamma,\alpha,T)\}}{\partial \hat{p}_n(\alpha)} \\ &= e^{-\beta t_n(x,\alpha,\gamma)} > 0\end{aligned}$$

□

Lemma 39. $\dfrac{dE\left\{g_n\left(x,-\gamma,\alpha,T\right)\right\}}{d\check{p}_n\left(\alpha\right)} = e^{-\beta t_n(x,\alpha,-\gamma)}$

Proof. By the definition of t_n we have

$$t_n\left(x,-\gamma,a\right) = c^{-1}\left|f^{-1}\left(\check{p}_n\left(a\right)\right) - f^{-1}\left(x\right)\right|$$

Differentiating with respect to $\check{p}_n\left(a\right)$ obtains

$$\frac{\partial t_n\left(x,-\gamma,a\right)}{\partial \check{p}_n\left(a\right)} = \frac{\operatorname{sgn}\left(\check{p}_n\left(a\right) - x\right)}{cf'\left(f^{-1}\left(\check{p}_n\left(a\right)\right)\right)}$$

By the definition of g_n we have

$$E\left\{g_n\left(x,-\gamma,\alpha,T\right)\right\} = \beta\int_0^{t_n(x,\alpha,-\gamma)} e^{-\beta t} f\left(f^{-1}\left(x\right) - \gamma t\right)dt + e^{-\beta t_n(x,\alpha,-\gamma)}\check{p}_n\left(\alpha\right)$$

Differentiating with respect to $t_n\left(x,\alpha,-\gamma\right)$ obtains

$$\frac{\partial E\left\{g_n\left(x,-\gamma,\alpha,T\right)\right\}}{\partial t_n\left(x,-\gamma,a\right)} = \beta e^{-\beta t_n(x,\alpha,-\gamma)}\check{p}_n\left(\alpha\right) - \beta e^{-\beta t_n(x,\alpha,-\gamma)}\check{p}_n\left(\alpha\right) = 0$$

Differentiating with respect to $\check{p}_n\left(\alpha\right)$ obtains

$$\begin{aligned}
&\frac{dE\left\{g_n\left(x,-\gamma,\alpha,T\right)\right\}}{d\check{p}_n\left(\alpha\right)} \\
= \;& \frac{\partial E\left\{g_n\left(x,-\gamma,\alpha,T\right)\right\}}{\partial t_n\left(x,-\gamma,a\right)}\frac{\partial t_n\left(x,-\gamma,a\right)}{\partial \check{p}_n\left(a\right)} + \frac{\partial E\left\{g_n\left(x,-\gamma,\alpha,T\right)\right\}}{\partial \check{p}_n\left(\alpha\right)} \\
= \;& e^{-\beta t_n(x,\alpha,-\gamma)} > 0
\end{aligned}$$

□

Lemma 40. $\hat{p}_n(\alpha)$ *is decreasing over* $\alpha \in (0,1)$ *for all* $n \in \mathbb{N}$.

Proof. By Lemma 24 we have $\hat{p}_0(\alpha)$ decreasing in α. For the inductive hypothesis, suppose $\hat{p}_n(\alpha)$ is decreasing in α. Then $E\{g_n(x,\gamma,\alpha,T)\}$ is also decreasing in α by Lemma 38. If $x \in (\bar{p}, \hat{p}_n(\alpha))$ then by the definition of g_n

$$E\{g_n(x,\gamma,\alpha,T)\} > x > \bar{p}$$

By the definition of λ_{n+1} we have

$$\lambda_{n+1}(x,\gamma,\alpha) = \alpha\bar{p} - x + (1-\alpha)E\{g_n(x,\gamma,\alpha,T)\}$$

Hence $\lambda_{n+1}(x,\gamma,\alpha)$ is decreasing over $\alpha \in (0,1)$ since $E\{g_n(x,\gamma,\alpha,T)\}$ is decreasing in α and $E\{g_n(x,\gamma,\alpha,T)\} > \bar{p}$. Now if $\alpha' \in (\alpha,1)$ then

$$\lambda_{n+1}(x,\gamma,\alpha') < \lambda_{n+1}(x,\gamma,\alpha) \quad \text{for all } x \in (\bar{p}, \hat{p}_n(\alpha))$$

Then by definition of $\hat{p}_{n+1}$ we have $\hat{p}_{n+1}(\alpha') \leq \hat{p}_{n+1}(\alpha)$. Therefore $\hat{p}_{n+1}(\alpha)$ is decreasing in α. □

Lemma 41. $\check{p}_n(\alpha)$ *is increasing over* $\alpha \in (0,1)$ *for all* $n \in \mathbb{N}$.

Proof. By Lemma 25 we have $\check{p}_0(\alpha)$ increasing in α. For the inductive hypothesis, suppose $\check{p}_n(\alpha)$ is increasing in α. Then $E\{g_n(x,-\gamma,\alpha,T)\}$ is also increasing in α by Lemma 39. If $x \in (\check{p}_n(\alpha), \bar{p})$ then by the definition of g_n

$$E\{g_n(x,-\gamma,\alpha,T)\} < x < \bar{p}$$

By the definition of λ_{n+1} we have

$$\lambda_{n+1}(x,-\gamma,\alpha) = \alpha\bar{p} - x + (1-\alpha)E\{g_n(x,-\gamma,\alpha,T)\}$$

Hence $\lambda_{n+1}(x,-\gamma,\alpha)$ is increasing over $\alpha \in (0,1)$ since $E\{g_n(x,-\gamma,\alpha,T)\}$ is increasing in α and $E\{g_n(x,-\gamma,\alpha,T)\} < \bar{p}$. Now if $\alpha' \in (\alpha,1)$ then

$$\lambda_{n+1}(x,-\gamma,\alpha') > \lambda_{n+1}(x,-\gamma,\alpha) \quad \text{for all } x \in (\check{p}_n(\alpha), \bar{p})$$

Then by definition of $\check{p}_{n+1}$ we have $\check{p}_{n+1}(\alpha') \geq \check{p}_{n+1}(\alpha)$. Therefore $\check{p}_{n+1}(\alpha)$ is increasing in α. □

Lemma 42. *$E\{g_n(x,\gamma,a,T)\}$ decreasing in a and $E\{g_n(x,-\gamma,a,T)\}$ increasing in a for all $n \in \mathbb{N}$ and $a \in (0,1)$.*

Proof. By Lemma 38 we can write

$$\frac{dE\{g_n(x,\gamma,a,T)\}}{d\hat{p}_n(a)} = e^{-\beta t_n(x,a,\gamma)} > 0$$

By Lemma 40 we have $\hat{p}_n(a)$ decreasing in a so $E\{g_n(x,\gamma,a,T)\}$ is also decreasing in α. By Lemma 39 we can write

$$\frac{dE\{g_n(x,-\gamma,\alpha,T)\}}{d\check{p}_n(\alpha)} = e^{-\beta t_n(x,\alpha,-\gamma)} > 0$$

By Lemma 41 we have $\check{p}_n(a)$ increasing in a so $E\{g_n(x,-\gamma,a,T)\}$ is also increasing in α. □

Lemma 43. $\lim_{\alpha\to 1} \hat{p}_n(\alpha) = \bar{p}$ *for all* $n \in \mathbb{N}$

Proof. By Lemma 26 we have $\lim_{\alpha\to 1} \hat{p}_0(\alpha) = \bar{p}$. For the inductive hypothesis, suppose that $\lim_{\alpha\to 1} \hat{p}_n(\alpha) = \bar{p}$. By the definition of $\hat{p}_{n+1}$ we can write

$$\begin{aligned}\hat{p}_{n+1}(\alpha) &= \inf\{x \in [\bar{p}, \hat{p}_n(\alpha)] : \lambda_{n+1}(x,\gamma,\alpha) < 0\} \\ \hat{p}_{n+1}(\alpha) &\geq \bar{p}\end{aligned}$$

By the definition of λ_{n+1} and Lemma 42 we have

$$\begin{aligned}\lambda_{n+1}(x,\gamma,\alpha) &= \alpha\bar{p} - x + (1-\alpha) E\{g_n(x,\gamma,\alpha,T)\} \\ \lim_{\alpha\to 1} \lambda_{n+1}(x,\gamma,\alpha) &= \bar{p} - x\end{aligned}$$

Hence for all $x > \bar{p}$ there exists $A(x) \in (0,1)$ such that

$$\begin{aligned}\alpha \in (A(x), 1) &\implies \lambda_{n+1}(x,\gamma,\alpha) < 0 \\ &\implies \hat{p}_{n+1}(\alpha) \in [\bar{p}, x]\end{aligned}$$

Thus $\lim_{\alpha\to 1} \hat{p}_{n+1}(\alpha) = \bar{p}$ by the squeeze theorem. □

Lemma 44. $\lim_{\alpha \to 1} \check{p}_n(\alpha) = \bar{p}$ *for all* $n \in \mathbb{N}$

Proof. By Lemma 27 we have $\lim_{\alpha \to 1} \check{p}_0(\alpha) = \bar{p}$. For the inductive hypothesis, suppose that $\lim_{\alpha \to 1} \check{p}_n(\alpha) = \bar{p}$. By the definition of $\check{p}_{n+1}$ we can write

$$\begin{aligned} \check{p}_{n+1}(a) &= \sup \{x \in [\check{p}_n(a), \bar{p}] : \lambda_{n+1}(x, -\gamma, a) > 0\} \\ \check{p}_{n+1}(\alpha) &\leq \bar{p} \end{aligned}$$

By the definition of λ_{n+1} and Lemma 42 we have

$$\begin{aligned} \lambda_{n+1}(x, -\gamma, \alpha) &= \alpha \bar{p} - x + (1 - \alpha) E\{g_n(x, -\gamma, \alpha, T)\} \\ \lim_{\alpha \to 1} \lambda_{n+1}(x, -\gamma, \alpha) &= \bar{p} - x \end{aligned}$$

Hence for all $x < \bar{p}$ there exists $A(x) \in (0, 1)$ such that

$$\begin{aligned} \alpha \in (A(x), 1) &\implies \lambda_{n+1}(x, -\gamma, \alpha) > 0 \\ &\implies \check{p}_{n+1}(\alpha) \in [x, \bar{p}] \end{aligned}$$

Thus $\lim_{\alpha \to 1} \check{p}_{n+1}(\alpha) = \bar{p}$ by the squeeze theorem. □

Lemma 45. $\lim_{\alpha \to 0} \hat{p}_n(\alpha) = \infty$ *for all* $n \in \mathbb{N}$

Proof. By Lemma 28 we have $\lim_{\alpha \to 0} \hat{p}_0(\alpha) = \infty$. For the inductive hypothesis suppose that $\lim_{\alpha \to 0} \hat{p}_n(\alpha) = \infty$. Then for all $y > \bar{p}$ there exists $K_n(y) \in (0,1)$ such that

$$\alpha \in (0, K_n(y)) \implies \hat{p}_n(\alpha) > y$$

By the definition of g_n for all $x \in (\bar{p}, \hat{p}_n(\alpha))$ we can write

$$E\{g_n(x, \gamma, T)\} > x \geq \bar{p}$$

Since $g_n(x, c, t)$ is continuous in x, it attains a minimum on every compact interval. So if $y > \bar{p}$ then there exists $A(y), B(y)$ such that

$$0 > A(y) = \min_{x \in [\bar{p}, y]} E\{g_n(x, \gamma, T)\} - x$$
$$0 < B(y) = \min_{x \in [\bar{p}, y]} \bar{p} - E\{g_n(x, \gamma, T)\}$$

By the definition of λ_{n+1}, for all $x \in [\bar{p}, y]$ we have

$$\lambda_{n+1}(x, \gamma, \alpha) = \alpha \bar{p} - x + (1 - \alpha) E\{g_n(x, \gamma, \alpha, T)\}$$
$$\lambda_{n+1}(x, \gamma, \alpha) = E\{g_n(x, \gamma, T)\} - x + \alpha [\bar{p} - E\{g_n(x, \gamma, T)\}]$$
$$\lambda_{n+1}(x, \gamma, \alpha) \geq A(y) + \alpha B(y)$$

Since $A(y) > 0$ then for all $y \in (\bar{p}, \hat{p}_n(\alpha))$ there exists $K_{n+1}(y) > 0$ such that

$$\alpha \in (0, K_{n+1}(y)) \implies \lambda_{n+1}(x, \gamma, \alpha) \geq 0 \quad \text{for all } x \in [\bar{p}, y]$$

Hence for all $y \in (\bar{p}, \hat{p}_n(\alpha))$ if $\alpha \in (0, K_{n+1}(y))$ then

$$\hat{p}_{n+1}(\alpha) = \inf\{x \in [\bar{p}, \hat{p}_n(\alpha)] : \lambda_{n+1}(x, \gamma, \alpha) < 0\} \geq y$$

Therefore $\lim_{\alpha \to 0} \hat{p}_{n+1}(\alpha) = \infty$ since $\lim_{\alpha \to 0} \hat{p}_n(\alpha) = \infty$. □

Lemma 46. $\lim_{\alpha \to 0} \check{p}_n(\alpha) = 0$ *for all* $n \in \mathbb{N}$

Proof. By Lemma 29 we have $\lim_{\alpha \to 0} \check{p}_0(\alpha) = 0$. For the inductive hypothesis suppose that $\lim_{\alpha \to 0} \check{p}_n(\alpha) = 0$. Then for all $y > 0$ there exists $K_n(y) \in (0,1)$ such that

$$\alpha \in (0, K_n(y)) \implies \hat{p}_n(\alpha) < y$$

If $x \in (\check{p}_n(\alpha), \bar{p})$ then by the definition of g_n we can write

$$E\{g_n(x, -\gamma, T)\} < x \leq \bar{p}$$

Since $g_n(x, c, t)$ is continuous in x, it attains a maximum on every compact interval. So if $y \in (\check{p}_n(\alpha), \bar{p})$ then there exists $A(y), B(y)$ such that

$$0 < A(y) = \max_{x \in [y, \bar{p}]} E\{g_n(x, -\gamma, T)\} - x$$
$$0 > B(y) = \max_{x \in [y, \bar{p}]} \bar{p} - E\{g_n(x, -\gamma, T)\}$$

Then by the definition of λ_{n+1}, for all $x \in [y, \bar{p}]$ we have

$$\lambda_{n+1}(x, -\gamma, \alpha) = \alpha\bar{p} - x + (1-\alpha) E\{g_n(x, -\gamma, \alpha, T)\}$$
$$\lambda_{n+1}(x, -\gamma, \alpha) = E\{g_n(x, -\gamma, T)\} - x + \alpha[\bar{p} - E\{g_n(x, -\gamma, T)\}]$$
$$\lambda_{n+1}(x, -\gamma, \alpha) \leq A(y) + \alpha B(y)$$

Since $A(y) < 0$ then for all $y \in (\check{p}_n(a), \bar{p})$ there exists $K_{n+1}(y) > 0$ such that

$$\alpha \in (0, K_{n+1}(y)) \implies \lambda_{n+1}(x, -\gamma, \alpha) < 0 \quad \text{for all } x \in [y, \bar{p}]$$

Hence for all $y \in (\check{p}_n(a), \bar{p})$ if $\alpha \in (0, K_{n+1}(y))$ then

$$\check{p}_{n+1}(a) = \sup\{x \in [\check{p}_n(a), \bar{p}] : \lambda_{n+1}(x, -\gamma, a) > 0\} \in (\check{p}_n(a), y)$$

Therefore $\lim_{\alpha \to 0} \check{p}_{n+1}(\alpha) = 0$ since $\lim_{\alpha \to 0} \check{p}_n(\alpha) = 0$. □